Embedding

To fully appreciate 2 sphere we embed it in 3D flat space

At least 5D is required to represent 4D spacetime

Alternative - take 2D slice of 4D spacetime
+ embed slice in 3D

Recall 2D space $ds^2 = a^2(d\theta^2 + \sin^2\theta\, d\phi^2)$

and 3D flat space $ds^2_{3flat} = d\rho^2 + \rho^2(d\psi^2 + \sin^2\psi\, d\varphi^2)$

in general, write $\rho = \rho(\theta,\phi)$; $\psi = \psi(\theta,\phi)$; $\varphi = \varphi(\theta,\phi)$

<u>an example</u> $ds^2 = -dt^2 + dr^2 + (b^2 + r^2)(d\theta^2 + \sin^2\theta\, d\phi^2)$

r can be pos or neg

- metric is curved
- at fixed r, t geometry is that of a 2-sphere with radius $r^2 + b^2$
- for $r \to \pm\infty$ $ds^2 \to ds^2_{flat}$ asymptotically flat

- spacetime is time-independent - focus on space part

$$ds^2_{3D} = dr^2 + (b^2+r^2)(d\theta^2 + \sin^2\theta \, d\phi^2)$$

- spherical symmetry means we can take any slice
e.g. $\theta = \pi/2$ - equatorial plane

$$d\Sigma^2 = dr^2 + (b^2+r^2)\, d\phi^2$$

embed in 3D flat space

$$ds^2_{3D\text{flat}} = d\rho^2 + \rho^2 d\varphi^2 + dz^2$$

use $\varphi = \phi$; $\rho = \rho(r)$; $z = z(r)$

$$ds^2_{\substack{3D\text{flat}\\ \text{on surface}}} = \left(\left(\frac{d\rho}{dr}\right)^2 + \left(\frac{dz}{dr}\right)^2\right) dr^2 + \rho^2(r)\, d\phi^2$$

so $\left(\frac{d\rho}{dr}\right)^2 + \left(\frac{dz}{dr}\right)^2 = 1 \qquad \rho^2 = r^2 + b^2$

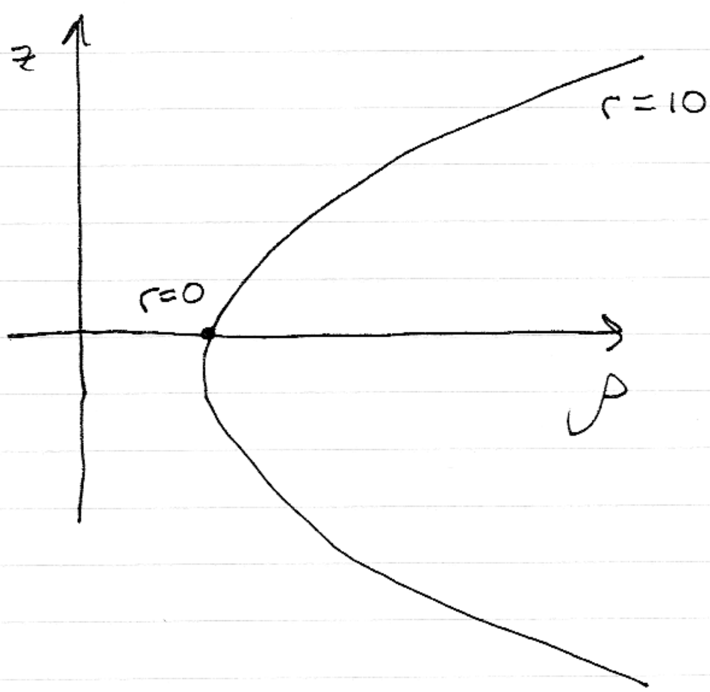
$$\Rightarrow \quad \rho\, d\rho = r\, dr \quad \text{so} \quad \frac{dz}{dr} = \left(1 - \frac{r^2}{\rho^2}\right)^{1/2}$$

$$= \frac{b}{(r^2+b^2)^{1/2}}$$

$$z = b \int \frac{dr}{(r^2+b^2)^{1/2}} = b \sinh^{-1}(r/b)$$

or $\quad r = b \sinh(z/b)$

$$\rho = (r^2+b^2)^{1/2} = b \cosh(z/b)$$



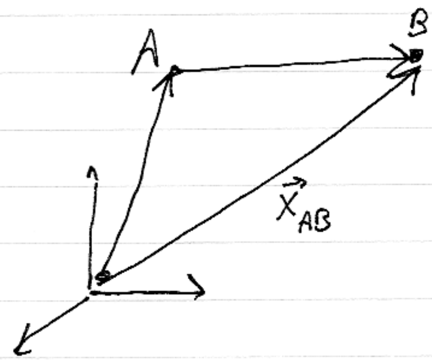
Vectors

1st year definition - quantity with magnitude + direction
obeys rules of addition, scalar products ...

e.g. displacement, velocity, acceleration

of these, displacement is different - non-local

$$\vec{X}_{AB} = \vec{X}_B - \vec{X}_A$$



$$\vec{v} = \left.\frac{d\vec{x}}{dt}\right|_{x=x_p,t}$$

$$\vec{a} = \left.\frac{d\vec{v}}{dt}\right|_{x=x_p,t}$$

In general relativity, only <u>local</u> vectors make sense

Can write vectors in terms of basis

$$\underset{\sim}{a}(x) = a^\alpha(x)\, \underset{\sim}{e}_\alpha(x) \qquad \alpha \text{ labels basis vectors}$$

$$(\underset{\sim}{e}_\alpha)^\beta \quad \text{is } \beta \text{ comp of } \underset{\sim}{e}_\alpha$$

Put another way

$$u^\alpha = \frac{dx^\alpha}{d\tau}$$ only means something if we specify basis

We know $\underset{\sim}{u} \cdot \underset{\sim}{u} = -1 = g_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau}$

$$= u^\alpha \underset{\sim}{e}_\alpha \, u^\beta \underset{\sim}{e}_\beta$$

$\Rightarrow \quad \underset{\sim}{e}_\alpha \cdot \underset{\sim}{e}_\beta = g_{\alpha\beta}$ by definition

Scalar product $\underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{e}_\alpha \cdot \underset{\sim}{e}_\beta \, a^\alpha b^\beta$

2 choices for basis

- orthonormal $\underset{\sim}{e}_{\hat\alpha} \cdot \underset{\sim}{e}_{\hat\beta} = \eta_{\hat\alpha\hat\beta}$

  i.e., 4 unit vectors

  think observer with $\underset{-}{u}_{obs} = \underset{\sim}{e}_{\hat t} = (1, 0, 0, 0)$ rest frame

  + 3 orthogonal unit vectors

  $\underset{\sim}{p}$ 4-momentum of part. in lab

  $p^{\hat t} = -\underset{\sim}{p} \cdot \underset{\sim}{u}_{obs} = E = \text{energy}$

- coordinate basis $\underset{\sim}{e}_\alpha \cdot \underset{\sim}{e}_\beta = g_{\alpha\beta}$

  $\underset{\sim}{a} \cdot \underset{\sim}{b} = g_{\alpha\beta} \, a^\alpha b^\beta$

  e.g. $\underset{\sim}{u} \cdot \underset{\sim}{u} = g_{\alpha\beta} u^\alpha u^\beta \equiv g_{\alpha\beta} \dfrac{dx^\alpha}{d\tau} \dfrac{dx^\beta}{d\tau}$ $u^\alpha \equiv \dfrac{dx^\alpha}{d\tau}$ in coordinate basis

  $$= \frac{g_{\alpha\beta} dx^\alpha dx^\beta}{(d\tau)^2} = -1$$