Energy of a particle or photon with 4-momentum $\mathbf{p}$ as measured by an observer with 4-velocity $\mathbf{u}_{\rm obs}$ is

$$E = -\mathbf{p} \cdot \mathbf{u}_{\rm obs} \tag{17}$$

For a particle, $\mathbf{p} = m\mathbf{u}$.

For a photon, $E = \hbar\omega = h\nu$.

COORDINATE AND ORTHONORMAL BASES

- A set $\{\mathbf{e}_{\hat{\alpha}}\}$ of four *orthonormal* basis vectors satisfies

$$\mathbf{e}_{\hat{\alpha}}(x) \cdot \mathbf{e}_{\hat{\beta}}(x) = \eta_{\hat{\alpha}\hat{\beta}}.$$

- A set $\{\mathbf{e}_\alpha\}$ of four *coordinate* basis vectors associated with a set of coordinates x^α satisfies

$$\mathbf{e}_\alpha(x) \cdot \mathbf{e}_\beta(x) = g_{\alpha\beta}(x)$$

where the line element has the form $ds^2 = g_{\alpha\beta}(x)dx^\alpha dx^\beta$.

- If the coordinate system is *orthogonal* ($g_{\alpha\beta}(x) = 0$ for $\alpha \neq \beta$), the coordinate basis components of an orthonormal basis pointing along the coordinate directions have the form

$$(\mathbf{e}_{\hat{0}})^\alpha = [(-g_{00})^{-1/2}, 0, 0, 0], \quad (\mathbf{e}_{\hat{1}})^\alpha = [0, (g_{11})^{-1/2}, 0, 0], \quad \text{etc.}$$

THE GEODESIC EQUATION

- Lagrangian for the Geodesic Equation of a test particle

$$L\left(\frac{dx^\alpha}{d\sigma}, x^\alpha\right) = \left(-g_{\alpha\beta}(x)\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma}\right)^{1/2}$$

where σ is an arbitrary parameter along the world line $x^\alpha = x^\alpha(\sigma)$ of the geodesic.

- Geodesic equation for a test particle (coordinate basis)

$$\frac{d^2x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} \quad \text{or} \quad \frac{du^\alpha}{d\tau} = -\Gamma^\alpha_{\beta\gamma}u^\beta u^\gamma$$

where τ is the proper time along the geodesic and $u^\alpha = dx^\alpha/d\tau$ are the coordinate basis components of the four-velocity so that $\mathbf{u} \cdot \mathbf{u} = -1$. The Christoffel symbols $\Gamma^\alpha_{\beta\gamma}$ follow from Lagrange's equations or from the general formula (8.19). The geodesic equation for light rays takes the same form with τ replaced by an affine parameter and $\mathbf{u} \cdot \mathbf{u} = 0$.

- Conserved Quantities

$$\boldsymbol{\xi} \cdot \mathbf{u} = \text{constant}$$

where $\boldsymbol{\xi}$ is a Killing vector, e.g., $\xi^\alpha = (0,\ 1,\ 0,\ 0)$ in a coordinate basis where the metric $g_{\alpha\beta}(x)$ is independent of x^1.

$$\Gamma^\alpha_{\gamma\epsilon} = \Gamma^\alpha_{\epsilon\gamma} = \frac{1}{2}g^{\alpha\beta}\left(\partial_\gamma g_{\beta\epsilon} + \partial_\epsilon g_{\beta\gamma} - \partial_\beta g_{\gamma\epsilon}\right) \tag{10}$$