PS4

1a) Schwarzschild

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

$$\simeq -\left(1+\frac{2\Phi}{c^2}\right)dt^2 + \left(1-\frac{2\Phi}{c^2}\right)dr^2 + r^2 d\Omega^2$$

usually, we have

$$dr^2 + r^2 d\Omega^2 = dx^2 + dy^2 + dz^2$$

where $r = (x^2+y^2+z^2)^{1/2}$ etc.

Here, we have

$$ds^2 = -\left(1+\frac{2\Phi}{c^2}\right)dt^2 + \left(1-\frac{2\Phi}{c^2}\right)(dx^2+dy^2+dz^2)$$

$$+ \frac{2\Phi}{c^2} r^2 d\Omega^2$$

extra term

$$r = r_1\left(1+\frac{1}{2r_1}\right)^2 \qquad \frac{GM}{c^2} = 1$$

$$dr = dr_1\left(\left(1+\frac{1}{2r_1}\right)^2 + 2\left(1+\frac{1}{2r_1}\right)\left(-\frac{1}{2r_1^2}\right)r_1\right)$$

$$= dr_1\left(\left(1+\frac{1}{2r_1}\right)\left(1+\frac{1}{2r_1}-\frac{1}{r_1}\right)\right)$$

$$= dr_1\left(1+\frac{1}{2r_1}\right)\left(1-\frac{1}{2r_1}\right)$$

$$ds^2 = -\left(1-\frac{2}{r}\right)dt^2 + \left(1-\frac{2}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

$$dr^2 = dr_1^2\left(1+\frac{1}{2r_1}\right)^2\left(1-\frac{1}{2r_1}\right)^2$$

$$1-\frac{2}{r} = 1-\frac{2}{r_1\left(1+\frac{1}{2r_1}\right)^2} = \left(\frac{1}{1+\frac{1}{2r_1}}\right)^2\left(\left(1+\frac{1}{2r_1}\right)^2-\frac{2}{r_1}\right)$$

$$= \left(\frac{1}{1+\frac{1}{2r_1}}\right)^2\left(1+\frac{1}{r_1}+\frac{1}{4r_1^2}-\frac{2}{r_1}\right)$$

$$= \left(1-\frac{1}{2r_1}\right)^2 \Big/ \left(1+\frac{1}{2r_1}\right)^{2}$$

so $$\left(1-\frac{2}{r}\right)^{-1} dr^2 = \frac{\left(1+\frac{1}{2r_1}\right)^2}{\cancel{\left(1-\frac{1}{2r_1}\right)^2}} \left(1+\frac{1}{2r_1}\right)^2 \cancel{\left(1-\frac{1}{2r_1}\right)^2} dr_1^2$$

$$= \left(1+\frac{1}{2r_1}\right)^4 dr_1^2$$

$$r^2 d\Omega^2 = \left(1+\frac{1}{2r_1}\right)^4 d\Omega^2$$

and $$\left(1-\frac{2}{r}\right) dt^2 = \frac{\left(1-\frac{1}{2r_1}\right)^2}{\left(1+\frac{1}{2r_1}\right)^2} dt^2$$

$$\Rightarrow ds^2 = -\frac{\left(1-\frac{1}{2r_1}\right)^2}{\left(1+\frac{1}{2r_1}\right)^2} dt^2 + \frac{\left(1+\frac{1}{2r_1}\right)^2}{\left(1-\frac{1}{2r_1}\right)^2} \left(dr_1^2 + r_1^2 d\Omega^2\right)$$

& since $dr_1^2 + r_1^2 d\Omega^2 = dx^2 + dy^2 + dz^2$

where $r_1^2 = x^2 + y^2 + z^2$ & $\tan\theta = \frac{(x^2+y^2)^{1/2}}{z}$

$\tan\phi = y/x$

we have

$$ds^2 = -\left(\frac{1 - 1/2r_1}{1 + 1/2r_1}\right)^2 dt^2 + \left(1 + 1/2r_1\right)^4 \left(dx^2 + dy^2 + dz^2\right)$$

for $\frac{1}{2r_1} \ll 1$ we have

$$ds^2 = -\left(1 - \frac{2}{r_1}\right) dt^2 + \left(1 + \frac{2}{r_1}\right) d\vec{x} \cdot d\vec{x}$$

$\frac{2}{r_1} \simeq \frac{2GM}{c^2 r_1} \simeq -\frac{2\Phi_{Newton}}{c^2}$ we have

Newtonian form

$$ds^2 = -\left(1 + 2\Phi/c^2\right) dt^2 + \left(1 - 2\Phi/c^2\right) d\vec{x} \cdot d\vec{x}$$

9.7 $E = -\underset{\sim}{u}_{obs} \cdot \underset{\sim}{p} = \gamma m = \dfrac{m}{(1-v^2)^{1/2}}$

or $\left(\dfrac{1}{1-v^2}\right)^{1/2} = -\underset{\sim}{u}_{obs} \cdot \underset{\sim}{u}$

$u^{\alpha}_{obs} = (u^{t}_{obs}, 0, 0, 0) \qquad u^{t}_{obs} = \left(1 - \dfrac{2M}{r}\right)^{-1/2}$

$\Rightarrow (1-v^2)^{1/2} = \dfrac{\left(1 - \frac{2M}{r}\right)^{1/2}}{e^2}$

$1 - v^2 = \dfrac{1}{e^2} - \dfrac{2M}{e^2 r} \qquad v = \left(1 - \dfrac{1}{e^2} + \dfrac{2M}{e^2 r}\right)^{1/2}$

set $r = 6M$ and $e = 1$ or 2

$V(e) = \left(1 - \dfrac{1}{e^2} + \dfrac{1}{3e^2}\right)^{1/2} = \left(1 - \dfrac{2}{3e^2}\right)^{1/2}$

$\dfrac{V(2)}{V(1)} = \left(\dfrac{1 - 1/6}{1 - 2/3}\right)^{1/2} = \left(\dfrac{5}{2}\right)^{1/2}$ ratio

or $V(2) = (5/6)^{1/2} \qquad V(1) = (1/3)^{1/2}$