9.17 Newtonian photons

$$L = r^2 \frac{d\phi}{dt}$$

$$\ddot{\vec{r}} = -\frac{GM}{r^2}\hat{r}$$

$$E = \frac{1}{2}\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}r^2\left(\frac{d\phi}{dt}\right)^2 - \frac{GM}{r}$$

$$\frac{1}{2}v^2 + \left(-\frac{GM}{r}\right) = E$$

$$= \frac{1}{2}\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}\frac{L^2}{r^2} - \frac{GM}{r}$$

with $\frac{dr}{dt} = 0$ as $r \to \infty$ $\Rightarrow$ $E = \frac{c^2}{2}$

$$\frac{dr}{dt} = \left(c^2 + \frac{2GM}{r} - \frac{L^2}{r^2}\right)^{1/2}$$

$$\frac{d\phi}{dr} = \frac{L}{r^2}\frac{1}{\left(c^2 + \frac{2GM}{r} - \frac{L^2}{r^2}\right)^{1/2}} = \frac{1}{r^2\left(\frac{c^2}{L^2} + \frac{2GM}{L^2 r} - \frac{1}{r^2}\right)^{1/2}}$$

$$\Delta\phi = 2\int_{r_{min}}^{\infty} \frac{dr/r^2}{\left(\frac{c^2}{L^2} + \frac{2GM}{L^2 r} - \frac{1}{r^2}\right)^{1/2}}$$

$$= 2\int_0^{u_{min}} \frac{du}{\left(c^2/L^2 + \frac{2GM}{L^2}u - u^2\right)^{1/2}}$$

$c = -1$ $\quad b = \frac{2GM}{L^2}$

$a = c^2/L^2$

$-\Delta = b^2 - 4ac$

$$\Delta\phi = \sin^{-1}\left(\frac{2u - \frac{2GM}{L^2}}{\left(\left(\frac{2GM}{L^2}\right)^2 + \frac{4c^2}{L^2}\right)^{1/2}}\right)_0^{u_{min}}$$

ignore M^2 term in denominator

$$= \sin^{-1}\left(\frac{Lu}{c} - \frac{GM}{cL}\right)_0^{u_{min}}$$

$$u_{min} = \frac{GM}{L^2} + \left(\left(\frac{GM}{L^2}\right)^2 + \frac{c^2}{L^2}\right)^{1/2} \approx \frac{c}{L} + \frac{GM}{L^2}$$

$$\boxed{cb = L}$$

$$\Delta\phi = \sin^{-1}\left(1 - \frac{GM}{c^2 b}\right) - \sin^{-1}\left(\frac{GM}{c^2 b}\right)$$

$$= \pi - \frac{2GM}{c^2 b}$$