Equation Sheet
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PHYS414/PHYS823
Faculty of Arts and Science
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$$\nabla_\alpha A_\beta = \partial_\alpha A_\beta - \Gamma^\delta_{\alpha\beta} A_\delta \tag{1}$$

$$\nabla_\alpha A^\beta = \partial_\alpha A^\beta + \Gamma^\beta_{\alpha\delta} A^\delta \tag{2}$$

$$\nabla_\alpha T_{\beta\gamma} = \partial_\alpha T_{\beta\gamma} - \Gamma^\delta_{\alpha\beta} T_{\delta\gamma} - \Gamma^\delta_{\alpha\gamma} T_{\beta\delta} \tag{3}$$

$$R_{\kappa\lambda\mu\nu} = -R_{\lambda\kappa\mu\nu} = -R_{\kappa\lambda\nu\mu} = R_{\mu\nu\lambda\kappa} \tag{4}$$

$$R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\delta\beta\gamma} = 0 \tag{5}$$

$$R^\alpha{}_{\beta\alpha\delta} = R_{\beta\delta} \tag{6}$$

$$R_{\alpha\beta} = R_{\beta\alpha} \tag{7}$$

$$R^\alpha{}_{\beta\gamma\delta} = \frac{\partial \Gamma^\alpha_{\beta\delta}}{\partial x^\gamma} - \frac{\partial \Gamma^\alpha_{\beta\gamma}}{\partial x^\delta} + \Gamma^\alpha_{\gamma\epsilon}\Gamma^\epsilon_{\beta\delta} - \Gamma^\alpha_{\delta\epsilon}\Gamma^\epsilon_{\beta\gamma} \tag{8}$$

$$\nabla_\alpha g_{\beta\gamma} = 0 \tag{9}$$

$$\Gamma^\alpha_{\gamma\epsilon} = \Gamma^\alpha_{\epsilon\gamma} = \frac{1}{2} g^{\alpha\beta} \left(\partial_\gamma g_{\beta\epsilon} + \partial_\epsilon g_{\beta\gamma} - \partial_\beta g_{\gamma\epsilon} \right) \tag{10}$$

Geodesic equation

$$u^\mu \nabla_\mu u^\nu = 0 \tag{11}$$

$$\frac{d^2 x^\nu}{d\tau^2} = -\Gamma^\nu_{\mu\kappa} \frac{dx^\mu}{d\tau}\frac{dx^\kappa}{d\tau} \quad \text{or} \quad \frac{du^\nu}{d\tau} = -\Gamma^\nu_{\mu\kappa} u^\mu u^\kappa \tag{12}$$

$$H(t) = \frac{1}{a}\frac{da}{dt} \tag{13}$$

$$\rho_{\rm crit} = \frac{3H_0^2}{8\pi G} \tag{14}$$

For a density today in matter, $\rho_{m,0}$ we can define the dimensionless density parameter

$$\Omega_m = \frac{\rho_{m,0}}{\rho_{\rm crit}} \tag{15}$$

Likewise, for other forms of matter.

Friedmann equation:

$$\left(\frac{1}{a}\frac{da}{dt}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{16}$$

where $k = -1, 0, 1$ for an open, flat, or closed FRW cosmology, respectively.

Energy of a particle or photon with 4-momentum $\mathbf{p}$ as measured by an observer with 4-velocity $\mathbf{u}_{\rm obs}$ is

$$E = -\mathbf{p} \cdot \mathbf{u}_{\rm obs} \tag{17}$$

For a particle, $\mathbf{p} = m\mathbf{u}$.

For a photon, $E = \hbar\omega = h\nu$.

Flat Spacetime

Cartesian Coordinates

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \equiv \eta_{\alpha\beta} dx^\alpha dx^\beta$$

Spatial Spherical Polar Coordinates

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

Static, Weak Field Metric

$$ds^2 = -(1 + 2\Phi(x^i))\, dt^2 + (1 - 2\Phi(x^i))(dx^2 + dy^2 + dz^2), \quad (\Phi(x^i) \ll 1).$$

Schwarzschild Geometry

Schwarzschild Coordinates

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Eddington–Finkelstein Coordinates

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Kruskal–Szekeres Coordinates

$$ds^2 = \frac{32M^3}{r} e^{-r/2M} \left(-dV^2 + dU^2\right) + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Friedman–Robertson–Walker Cosmological Models

$$ds^2 = -dt^2 + a^2(t) \left[d\chi^2 + \left\{ \begin{matrix} \sin^2\chi \\ \chi^2 \\ \sinh^2\chi \end{matrix} \right\} (d\theta^2 + \sin^2\theta d\phi^2) \right], \quad \left\{ \begin{matrix} \text{closed} \\ \text{flat} \\ \text{open} \end{matrix} \right\}.$$

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right], \quad \left(\begin{matrix} k = +1, \text{ closed} \\ k = 0, \text{ flat} \\ k = -1, \text{ open} \end{matrix} \right).$$

THE GEODESIC EQUATION

- Lagrangian for the Geodesic Equation of a test particle

$$L\left(\frac{dx^\alpha}{d\sigma}, x^\alpha\right) = \left(-g_{\alpha\beta}(x) \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma}\right)^{1/2}$$

  where σ is an arbitrary parameter along the world line $x^\alpha = x^\alpha(\sigma)$ of the geodesic.
- Geodesic equation for a test particle (coordinate basis)

$$\frac{d^2x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau} \quad \text{or} \quad \frac{du^\alpha}{d\tau} = -\Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma$$

  where τ is the proper time along the geodesic and $u^\alpha = dx^\alpha/d\tau$ are the coordinate basis components of the four-velocity so that $\mathbf{u} \cdot \mathbf{u} = -1$. The Christoffel symbols $\Gamma^\alpha_{\beta\gamma}$ follow from Lagrange's equations or from the general formula (8.19). The geodesic equation for light rays takes the same form with τ replaced by an affine parameter and $\mathbf{u} \cdot \mathbf{u} = 0$.
- Conserved Quantities

$$\boldsymbol{\xi} \cdot \mathbf{u} = \text{constant}$$

  where $\boldsymbol{\xi}$ is a Killing vector, e.g., $\xi^\alpha = (0, 1, 0, 0)$ in a coordinate basis where the metric $g_{\alpha\beta}(x)$ is independent of x^1.

COORDINATE AND ORTHONORMAL BASES

- A set $\{\mathbf{e}_{\hat{\alpha}}\}$ of four *orthonormal* basis vectors satisfies

$$\mathbf{e}_{\hat{\alpha}}(x) \cdot \mathbf{e}_{\hat{\beta}}(x) = \eta_{\hat{\alpha}\hat{\beta}}.$$

- A set $\{\mathbf{e}_{\alpha}\}$ of four *coordinate* basis vectors associated with a set of coordinates x^{α} satisfies

$$\mathbf{e}_{\alpha}(x) \cdot \mathbf{e}_{\beta}(x) = g_{\alpha\beta}(x)$$

  where the line element has the form $ds^2 = g_{\alpha\beta}(x)dx^{\alpha}dx^{\beta}$.
- If the coordinate system is *orthogonal* ($g_{\alpha\beta}(x) = 0$ for $\alpha \neq \beta$), the coordinate basis components of an orthonormal basis pointing along the coordinate directions have the form

$$(\mathbf{e}_{\hat{0}})^{\alpha} = [(-g_{00})^{-1/2}, 0, 0, 0], \quad (\mathbf{e}_{\hat{1}})^{\alpha} = [0, (g_{11})^{-1/2}, 0, 0], \quad \text{etc.}$$

USEFUL NUMBERS

Conversion Factors

Velocity of light	$c \equiv 299792458$ m/s $\approx 3 \times 10^{10}$ cm/s
Boltzmann's constant	$k_B = 1.38 \times 10^{-16}$ erg/K $= 8.59 \times 10^{-5}$ eV/K
Second of arc	1 arcsec = $1''$ =4.85×10^{-6} rad
Light year	1 ly = 9.46×10^{17} cm
Parsec	1 pc = 3.09×10^{18} cm = 3.26 ly
Electron volt	1 eV = 1.60×10^{-12} erg = 1.16×10^{4} K
Erg (cgs unit of energy)	1 erg = 10^{-7} J
Dyne (cgs unit of force)	1 dyne = 10^{-5} N

Physical Constants

Gravitational constant	$G = 6.67 \times 10^{-8}$ dyn $\cdot$ cm^2/g^2
Stefan–Boltzmann constant	$\sigma = 5.67 \times 10^{-5}$ erg/(cm$^2 \cdot$ s $\cdot$ K^4)
Radiation constant	$a = 7.56 \times 10^{-15}$ erg/(cm$^3 \cdot$ K^4)
Mass of an electron	$m_e = 9.11 \times 10^{-28}$ g
Mass of a proton	$m_p = 1.67 \times 10^{-24}$ g
Planck's constant	$\hbar = 1.05 \times 10^{-27}$ erg $\cdot$ s

ASTRONOMICAL CONSTANTS

Earth

Astronomical unit (semimajor axis of Earth's orbit)	$AU = 1.50 \times 10^{8}$ km
	$= 1.50 \times 10^{13}$ cm
Mass of the Earth	$M_{\oplus} = 5.97 \times 10^{27}$ g
	$GM_{\oplus}/c^2 = 0.443$ cm
Equatorial radius of the Earth	$R_{\oplus} = 6.38 \times 10^{8}$ cm $= 6378$ km
Moment of inertia about rotation axis	8.04×10^{44} g $\cdot$ cm$^2 = .331\ M_{\oplus} R_{\oplus}^2$
Rotation period	8.62×10^{4} s
Angular velocity	$\Omega_{\oplus} = 7.29 \times 10^{-5}$ rad/s

Sun

Mass of the Sun	$M_{\odot} = 1.99 \times 10^{33}$ g
	$GM_{\odot}/c^2 = 1.48$ km
Radius of the Sun	$R_{\odot} = 6.96 \times 10^{10}$ cm $= 6.96 \times 10^{5}$ km
Moment of inertia about rotation axis	5.7×10^{53} g $\cdot$ cm^2
Rotation period at Equator	25.5 days
Angular velocity at Equator	2.85×10^{-6} rad/s
Luminosity of the Sun	$L_{\odot} = 3.85 \times 10^{33}$ erg/s

Moon

Radius of the Moon's orbit (mean)	3.84×10^{5} km
Mass of the Moon	$M_{\mathrm{Moon}} = 7.35 \times 10^{25}$ g $= M_{\oplus}/81.3$
Radius of the Moon	$R_{\mathrm{Moon}} = 1.74 \times 10^{3}$ km

Our Galaxy (The Milky Way)

Mass of the Milky Way in visible matter	$\approx 10^{11} M_{\odot}$
Radius of the luminous Milky Way disk	$\approx 20 - 25$ kpc
Luminosity of the Milky Way	$\approx 4 \times 10^{10} L_{\odot}$

Universe

Hubble Constant	$H_0 \approx (72 \pm 7)[(\mathrm{km/s})/\mathrm{Mpc}]$
	$h \equiv H_0/(100\ [(\mathrm{km/s})/\mathrm{Mpc}]) \approx .7 \pm .1$
Hubble Time	$t_H \equiv H_0^{-1} = 9.78 \times 10^{9}\ h^{-1}$ yr
Hubble Distance	$d_H \equiv cH_0^{-1} = 2998\ h^{-1}$ Mpc
Critical density	$\rho_c \equiv 3H_0^2/8\pi G = 1.88 \times 10^{-29}\ h^2$ g/cm^3
Temperature of CMB today	$= 2.73$K

APPENDIX B

Curvature Quantities

The following tables give useful quantities for the simplest of the geometries considered in the text. Specifically they give the metric, Christoffel symbols, Riemann curvature, and Einstein curvature. These are enough to form the geodesic equations and the Einstein equation.

Only nonzero components are shown, and only those nonzero components sufficient to construct the rest by summetries. For instance, we don't give both Γ^t_{tr} and Γ^t_{rt}, since $\Gamma^\alpha_{\beta\gamma}$ is symmetric in β and γ. Similarly other nonzero components of the Riemann curvature can be found from the ones displayed by making use of the symmetries in (21.29).

In each case one coordinate system is used and the Christoffel symbols are given in that coordinate basis. Both the Christoffel symbols and coordinate basis components of the curvature quantities can be computed using the *Mathematica* program *Curvature and the Einstein Equation* on the book website. However, curvature quantities are quoted in an orthonormal basis. This gives simpler expressions for highly symmetric metrics, and ones that are not singular at coordinate singularities. Since all the metrics considered are diagonal, we use an orthonormal basis whose vectors point along the coordinate directions. The coordinate components of these basis vectors are easily calculated from the metric according to the prescription in Example 7.9. Specifically,

$$(\mathbf{e}_{\hat{0}})^\alpha = [(-g_{00})^{-1/2}, 0, 0, 0], \qquad \qquad \begin{pmatrix} \text{diagonal} \\ \text{metrics} \end{pmatrix}$$
$$(\mathbf{e}_{\hat{1}})^\alpha = [0, (g_{11})^{-1/2}, 0, 0], \quad \text{etc.}$$

Components in this coordinate basis and the orthonormal basis are connected by (20.41), which in the case of the Einstein curvature reads

$$G_{\hat{\alpha}\hat{\beta}} = (e_{\hat{\alpha}})^\alpha \, (e_{\hat{\beta}})^\beta \, G_{\alpha\beta}$$

which, for these simple diagonal metrics, reduces to a simple prescription, e.g.,

$$G_{\hat{0}\hat{1}} = (-g_{00})^{-1/2} \, G_{01} (g_{11})^{-1/2} \text{, etc.} \qquad \text{(diagonal metrics).}$$

The analogous relation for the Riemann curvature is given in (21.25). Inverting these relations allows the coordinate basis components to be computed from the orthonormal basis components given.

The Ricci curvature components and the Ricci curvature scalar can be found from the Riemann curvature components in an orthonormal basis by

$$R_{\hat{\alpha}\hat{\beta}} = \eta^{\hat{\gamma}\hat{\delta}} R_{\hat{\alpha}\hat{\gamma}\hat{\beta}\hat{\delta}}, \quad R = \eta^{\hat{\alpha}\hat{\beta}} R_{\hat{\alpha}\hat{\beta}}.$$

Schwarzschild Geometry

- **Metric (Schwarzschild coordinates):**

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

- **Christoffel Symbols:**

$\Gamma^t_{tr} = (M/r^2)(1-2M/r)^{-1}$

$\Gamma^r_{tt} = (M/r^2)(1-2M/r)$

$\Gamma^r_{rr} = -(M/r^2)(1-2M/r)^{-1}$

$\Gamma^r_{\theta\theta} = -(r-2M)$

$\Gamma^r_{\phi\phi} = -(r-2M)\sin^2\theta$

$\Gamma^\theta_{r\theta} = 1/r$

$\Gamma^\theta_{\phi\phi} = -\cos\theta\,\sin\theta$

$\Gamma^\phi_{r\phi} = 1/r$

$\Gamma^\phi_{\theta\phi} = \cot\theta$

- **An Orthonormal Basis:**

$(e_{\hat{t}})^\alpha = \left[(1-2M/r)^{-1/2}, 0, 0, 0\right]$

$(e_{\hat{r}})^\alpha = \left[0, (1-2M/r)^{1/2}, 0, 0, 0\right]$

$(e_{\hat{\theta}})^\alpha = [0, 0, 1/r, 0]$

$(e_{\hat{\phi}})^\alpha = [0, 0, 0, 1/(r\sin\theta)]$

- **Riemann Curvature:**

$R_{\hat{t}\hat{r}\hat{t}\hat{r}} = -2M/r^3$

$R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = +2M/r^3$

$R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}} = R_{\hat{t}\hat{\phi}\hat{t}\hat{\phi}} = +M/r^3$

$R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} = R_{\hat{r}\hat{\phi}\hat{r}\hat{\phi}} = -M/r^3$

- **Einstein Curvature**

$G_{\hat{\alpha}\hat{\beta}} = 0$

Friedman-Robertson-Walker (FRW) Geometries

- **Metric:**

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

- **Christoffel Symbols:**

$\Gamma^t_{rr} = a\dot{a}/(1-kr^2)$

$\Gamma^t_{\theta\theta} = r^2 a\dot{a}$

$\Gamma^t_{\phi\phi} = r^2 \sin^2\theta\, a\dot{a}$

$\Gamma^r_{rr} = kr/(1-kr^2)$

$\Gamma^r_{\theta\theta} = -r(1-kr^2)$

$\Gamma^r_{\phi\phi} = -r(1-kr^2)\, \sin^2\theta$

$\Gamma^r_{tr} = \dot{a}/a$

$\Gamma^\theta_{r\theta} = 1/r$

$\Gamma^\theta_{\phi\phi} = -\cos\theta\, \sin\theta$

$\Gamma^\theta_{t\theta} = \dot{a}/a$

$\Gamma^\phi_{r\phi} = 1/r$

$\Gamma^\phi_{\theta\phi} = \cot\theta$

$\Gamma^\phi_{t\phi} = \dot{a}/a$

- **An Orthonormal Basis:**

$(e_{\hat{t}})^\alpha = [1, 0, 0, 0]$

$(e_{\hat{r}})^\alpha = [0, \sqrt{1-kr^2}, 0, 0]/a$

$(e_{\hat{\theta}})^\alpha = [0, 0, 1/r, 0]/a$

$(e_{\hat{\phi}})^\alpha = [0, 0, 0, 1/(r\sin\theta)]/a$

- **Riemann Curvature:**

$R_{\hat{t}\hat{r}\hat{t}\hat{r}} = R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}} = R_{\hat{t}\hat{\phi}\hat{t}\hat{\phi}} = -\ddot{a}/a$

$R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} = R_{\hat{r}\hat{\phi}\hat{r}\hat{\phi}} = R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = (k+\dot{a}^2)/a^2$

- **Einstein Curvature:**

$G_{\hat{t}\hat{t}} = 3\left(k+\dot{a}^2\right)/a^2$

$G_{\hat{r}\hat{r}} = G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}} = -\left(k+\dot{a}^2+2a\ddot{a}\right)/a^2$