# Physics 414/823 Midterm; October 25, 2010

1. **(10 points)** Consider a spacetime described by the metric

$$ds^2 = -(1-\kappa|z|)^2 \, dt^2 + dz^2 + (1-\kappa|z|)^2 \left(dx^2 + dy^2\right) \tag{1}$$

where $\kappa$ is a positive constant.

(a) Write down the Killing vectors and conserved quantities for a massive particle (there are three Killing vectors!). What do the conserved quantities correspond to?

(b) A particle begins from rest at $z = 0$ and proper time $\tau = 0$ and starts to move in the positive $z$ direction. (You may assume that the particle starts with a negligibly small velocity in the positive $z$-direction.) Find $z$ as a function of the particle's proper time $\tau$.

(c) Work out the limiting form for $z(\tau)$ at early times. Be sure to define what is meant by "early times". What first-year physics problem does this motion remind you of? Could you have guessed this result from the form of the metric?

(d) At what proper time does the particle reach $\kappa =^{-1}$?

2. **(10 points)**

Again consider the spacetime described by Equation (1). Two observers are at rest in the above coordinate system. Observer A is at position $x = y = z = 0$ while Observer B is at position $x = y = 0,\ z = z_B$.

(a) Draw the lightcone structure for the metric in the $z - t$ plane.

(b) Write out the four-velocities for the two observers in component notation.

(c) Observer A sends a light signal with frequency $\omega_A$ to B. What frequency will B measure? Assume $0 < z_B < \kappa^{-1}$. What happens to $\omega_B$ as $z_B$ approaches $\kappa^{-1}$?

(d) Suppose B is replaced by a mirror which reflects the light signal back to A. According the A's clock, how much time elapses between her sending and receiving the signal?

(e) Discuss your the result from the part (d) for the case where observer B is close to observer A (you'll have to define what you mean by close) and for the case where $z_B$ approaches $\kappa^{-1}$.

1

a) $t$ independence $\quad \xi^\alpha = (1,0,0,0) \quad e = -\underset{\sim}{\xi} \cdot \underset{\sim}{u}$

1/2 energy $\quad$ +1 for correspondence $\quad = (1-K|z|)^2 \dfrac{dt}{d\tau}$

$x$ independence $\quad \eta_x^\alpha = (0,1,0,0) \quad p_x = \eta_x \cdot \underset{\sim}{u}$

1/2 $x$-momentum $\quad = (1-K|z|)^2 \dfrac{dx}{d\tau}$

$y$ independence $\quad \eta_y^\alpha = (0,0,1,0) \quad p_y = \eta_y \cdot \underset{\sim}{u}$

1/2 $y$ momentum $\quad = (1-K|z|)^2 \dfrac{dy}{d\tau}$

b) $\tau = 0 \qquad u^x = u^y = u^z = 0 \quad$ so $p_x = p_y = 0$

$$-1 = -(1-K|z|)^2\left(\frac{dt}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2$$

1/2

$$= -e^2 \quad \text{at } z=0,\ \frac{dz}{d\tau} = 0$$

so $e = 1$

$$\left(\frac{dz}{d\tau}\right)^2 = \frac{1}{(1-K|z|)^2} - 1$$

so $$\frac{dz}{\left(\frac{1}{(1-K|z|)^2} - 1\right)^{1/2}} = d\tau$$

$$\tau = \frac{1}{K}\left(1 - (1-K|z|)^2\right)^{1/2}$$ check $z=0$ $\tau=0$

$$(1-K|z|)^2 = 1 - K^2\tau^2$$

$$z = \frac{1}{K}\left(1 - (1-K^2\tau^2)^{1/2}\right)$$

c) $$z \simeq \frac{1}{2}K\tau^2$$

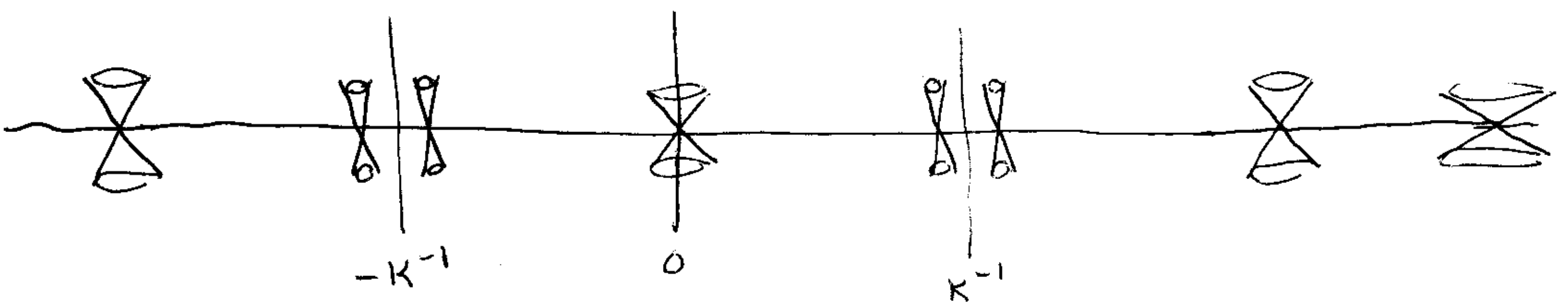
so uniform acceleration
with $\phi_{Newton} = K|z|$
$1\frac{1}{2}$
or constant force

d) $z = K^{-1}$ $\tau = K^{-1}$

$2^{1/2}$

2a) $$\frac{dt}{dz} = \pm\left(1 - K|z|\right)^{-1}$$

$$= \begin{cases} \pm 1 & z = 0 \\ \pm\infty & z \to K^{-1} \\ \pm 1 & |z| = 2K^{-1} \end{cases}$$




b) $$u^\alpha = (u^t, 0, 0, 0)$$

$$\underset{\sim}{u} \cdot \underset{\sim}{u} = -1 = -\left(1 - K|z|\right)^2 (u^t)^2 \qquad u^t = \left(1 - K|z|\right)^{-1}$$

c) for photon $\quad E = -\underset{\sim}{p} \cdot \underset{\sim}{u}_{obs}$

but $\quad e = -\underset{\sim}{\xi} \cdot \underset{\sim}{p} \quad$ is conserved

$$\underset{\sim}{u}_{obs} = \left(1 - K|z|\right)^{-1} \underset{\sim}{\xi}$$

so $\quad E = \left(1 - K|z|\right)^{-1} \times (\text{conserved quantity})$

$$\frac{E_B}{E_A} = \frac{\omega_B}{\omega_A} = \left(1 - \kappa|z_B|\right)^{-1}$$ blue shift

$\to \infty$ as $z \to \kappa^{-1}$

d) want $z = z(t)$ for A to B then double

$$\Delta\tau\Big)_{z=0} = \Delta t$$

$$\frac{dz}{dt} = 1 - \kappa|z| \qquad \frac{dz}{1-\kappa|z|} = dt$$

$$\Delta t = -\frac{2}{\kappa}\ln\left(1-\kappa z_B\right)$$

e) for $\kappa z_B \ll 1$ $\quad \Delta t \simeq 2z_B$ flat space result