3. (15 pts) In a simple model for the Universe, the number of galaxies is constant and galaxies are at fixed co-moving coordinates. Let $N(t)$ be the number density of galaxies (number of galaxies per physical volume). As discussed in class, $N(t)$ is the time-component of a number 4-vector N^α which satisfies the conservation law $\nabla_\alpha N^\alpha = 0$.

   Use this conservation law and the Christoffel symbols for the FRW cosmologies to derive the result:

$$N(t) = N_0 \left(\frac{a_0}{a(t)} \right)^3 \tag{8}$$

   where the subscript '0' refers to the present epoch.

4. (20 pts) The goal of this problem is to derive an expression for the number of galaxies with redshift less than a given value z. Recall that the redshift is related to the scale factor through the expression $a_0/a(t) = 1+z$. Assume that the number density of galaxies is given Eq. (8). That is, assume that galaxies are neither created nor destroyed between z and the present epoch. (You may use Eq.(8) even if you didn't derive it.)

   Use the metric for the FRW Universe models in the form given on the equation sheet. The observer is at $r = 0$ and distant galaxy at $r = r_e$ emits light at time t_e which is then detected by the observer at time t_0.

   (a) Write down an expression for the number of galaxies in a spherical shell of thickness dr_e and coordinate radius r_e.

   (b) Derive the relation between the coordinate r_e and the time t_e for the flat ($k = 0$) FRW cosmology. Recall that in the flat FRW cosmology, $a(t) = a_0\,(t/t_0)^{2/3}$.

   (c) Integrate this expression to derive the desired expression for the total number of galaxies with redshift less than z as a function of z.

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right)$$

a)

Volume of spherical shell

$$dV = a^3 \frac{dr}{(1-kr^2)^{1/2}} \times r^2 \times 4\pi$$

$$dN = N(t)\, dV = 4\pi N_0 a_0^3 \frac{dr_e}{(1-kr_e^2)^{1/2}} r_e^2$$

b) Along photon geodesic from t_e, r_e to $t_0, r=0$

we have $dt = a(t)\, dr$

or $$r_e = \int \frac{dt}{a(t)} = \int \frac{dt}{da} \frac{1}{a}\, da$$

+ use $\frac{da}{dt}$ from Freidmann equation

Here, we're given $a(t) = a_0 \left(t/t_0\right)^{2/3}$

$$r_e = \int_{t_e}^{t_o} \frac{dt}{a_o \left(t/t_o\right)^{2/3}} = \frac{t_o}{a_o} \int_{t_e/t_o}^{1} \frac{ds}{s^{2/3}}$$

$$= \frac{3t_o}{a_o} s^{1/3} \Big|_{t_e/t_o}^{1} = \frac{3t_o}{a_o} \left(1 - \left(t_e/t_o \right)^{1/3} \right)$$

$$dN = 4\pi N_o a_o^3 r_e^2 \, dr_e$$

$$= 4\pi N_o \cancel{a_o^3} \frac{9t_o^2}{\cancel{a_o^2}} \left(1 - \left(t_e'/t_o \right)^{1/3} \right)^2 \frac{dt_e'}{\cancel{a_o} \left(t_e'/t_o \right)^{2/3}}$$

$u = t_e'/t_o$

$$= 36\pi N_o t_o^3 \left(u^{-2/3} - 2u^{-1/3} + 1 \right) du$$

$$N(t_e) = 36\pi N_o t_o^3 \int_{t_e/t_o}^{1} \left(u^{-2/3} - 2u^{-1/3} + 1 \right) du$$

$$= 36\pi N_o t_o^3 \left(3u^{1/3} - 3u^{2/3} + u \right)_{t_e/t_o}^{1}$$

$$= 36\pi N_o t_o^3 \left(1 - 3\left(t_e/t_o \right)^{1/3} + 3\left(t_e/t_o \right)^{2/3} - t_e/t_o \right)$$

Now $\frac{t_e}{t_0} = \left(\frac{a_e}{a_0}\right)^{3/2} = (1+z)^{-3/2}$

$$N(z) = 36\pi N_0 t_0^3 \left(1 - 3(1+z)^{-1/2} + 3(1+z)^{-1} - (1+z)^{-3/2}\right)$$

at small z $\simeq \frac{9\pi}{2} N_0 t_0^3 z^3$

& with $1+z = \frac{a_0}{a} = \left(\frac{t_0}{t}\right)^{2/3}$ or $\frac{t_0}{t} = (1+z)^{3/2} \simeq 1 + \frac{3}{2}z$

$$d = t_0 - t = t_0(1 - t/t_0) \simeq t_0 \frac{3}{2} z$$

so $N(z) = \frac{4\pi}{3} N_0 d^3$

1. Any vector $\mathbf{w}$ satisfies the following equation.

$$\left(\nabla_\delta \nabla_\gamma - \nabla_\gamma \nabla_\delta\right) w_\beta = w^\alpha R_{\alpha\beta\gamma\delta} \tag{1}$$

where $R_{\beta\alpha\gamma\delta}$ is the Reimann curvature tensor. Consider a Killing vector ξ^α. In addition to Equation 1, ξ^α also satisfies the Killing equation

$$\nabla_\gamma \xi_\beta + \nabla_\beta \xi_\gamma = 0 \tag{2}$$

(a) Show that ξ^α satisfies the following two equations.

$$\nabla_\gamma \xi^\gamma = 0 \tag{3}$$

and

$$\nabla_\alpha \nabla^\alpha \xi_\gamma = -R_{\gamma\alpha} \xi^\alpha \tag{4}$$

(b) Define the rank-2 tensor $F_{\alpha\beta} \equiv \nabla_\alpha \xi_\beta - \nabla_\beta \xi_\alpha$. Show that F satisfies the equation

$$\nabla_\alpha F_{\beta\gamma} + \nabla_\gamma F_{\alpha\beta} + \nabla_\beta F_{\gamma\alpha} = 0 \tag{5}$$

You will find useful formulae on the equation sheet.

$$\nabla_\alpha A_\beta = \partial_\alpha A_\beta - \Gamma^\delta_{\alpha\beta} A_\delta \tag{1}$$

$$\nabla_\alpha T_{\beta\gamma} = \partial_\alpha T_{\beta\gamma} - \Gamma^\delta_{\alpha\beta} T_{\delta\gamma} - \Gamma^\delta_{\alpha\gamma} T_{\beta\delta} \tag{2}$$

$$R_{\kappa\lambda\mu\nu} = -R_{\lambda\kappa\mu\nu} = -R_{\kappa\lambda\nu\mu} = R_{\lambda\kappa\nu\mu} \tag{3}$$

$$R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\delta\beta\gamma} = 0 \tag{4}$$

$$R^\alpha{}_{\beta\alpha\delta} = R_{\beta\delta} \tag{5}$$

$$R_{\alpha\beta} = R_{\beta\alpha} \tag{6}$$

$$R^\alpha{}_{\beta\gamma\delta} = \frac{\partial \Gamma^\alpha_{\beta\delta}}{\partial x^\gamma} - \frac{\partial \Gamma^\alpha_{\beta\gamma}}{\partial x^\delta} + \Gamma^\alpha_{\gamma\epsilon} \Gamma^\epsilon_{\beta\delta} - \Gamma^\alpha_{\delta\epsilon} \Gamma^\epsilon_{\beta\gamma} \tag{7}$$

$$\nabla_\alpha g_{\beta\gamma} = 0 \tag{8}$$

$$\Gamma^\alpha_{\gamma\epsilon} = \Gamma^\alpha_{\epsilon\gamma} \tag{9}$$

a) $\nabla_\gamma \xi_\beta + \nabla_\beta \xi_\gamma = 0$

$$g^{\gamma\beta}\left(\nabla_\gamma \xi_\beta + \nabla_\beta \xi_\gamma\right) = 2\nabla^\gamma \xi_\gamma = 2\nabla_\gamma \xi^\gamma = 0$$

use $\nabla_\gamma g^{\gamma\beta} = 0$

b) $(\nabla_\delta \nabla_\gamma - \nabla_\gamma \nabla_\delta)\xi_\beta = \xi^\alpha R_{\alpha\beta\gamma\delta}$

But $\nabla_\delta \xi_\beta = -\nabla_\beta \xi_\delta$ so

$$g^{\beta\gamma}\left(\nabla_\delta \nabla_\gamma \xi_\beta + \nabla_\gamma \nabla_\beta \xi_\delta = \xi^\alpha R_{\alpha\beta\gamma\delta}\right)$$

+ we get $\underbrace{\nabla_\delta \cancel{\nabla^\beta} \xi_\beta}_{0} + \nabla^\beta \nabla_\beta \xi_\delta = \xi^\alpha g^{\beta\gamma} R_{\alpha\beta\gamma\delta}$

$$= -\xi^\alpha g^{\beta\gamma} R_{\beta\alpha\gamma\delta}$$

$$= -\xi^\alpha R_{\alpha\delta} \checkmark$$

$$\nabla_\alpha \overset{1}{\nabla_\beta} \xi_\gamma - \nabla_\alpha \overset{2}{\nabla_\gamma} \xi_\beta + \overset{2}{\nabla_\gamma \nabla_\alpha} \xi_\beta - \nabla_\gamma \overset{3}{\nabla_\beta} \xi_\alpha + \nabla_\beta \overset{3}{\nabla_\gamma} \xi_\alpha - \nabla_\beta \overset{1}{\nabla_\alpha} \xi_\gamma$$

$$= \xi^\lambda \overset{1}{R}_{\lambda\gamma\beta\alpha} + \xi^\lambda R_{\lambda\beta\alpha\gamma} + \xi^\lambda R_{\lambda\alpha\gamma\beta} = 0$$

3. Astronauts wish to orbit the supermassive black hole at the center of the galaxy. Assume the metric is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{9}$$

They decide to follow a circular orbit with $r = R$, $\theta = \pi/2$ and angular velocity $d\phi/dt = \Omega$ where Ω is a constant. Now if they choose the freely orbiting value for Ω, they will experience weightlessness and their bones and muscles will atrophy. They decide instead to pick a non-freely falling orbit.

(a) Calculate the four-acceleration, $a^\mu = u^\nu \nabla_\nu u^\mu$ and its magnitude $a = (a^\mu a_\mu)^{1/2}$ as a function of Ω, R and M. For what value of Ω does $a = 0$?

(b) Suppose the astronauts wish to experience an Earth-like "force". Set $a = g$ and solve for Ω. Note that there are two solutions.

(c) Find the two values of Ω for $R = 10M$, $a = g = 9.8\,\mathrm{m\,s^{-2}}$, and $M = 10^9 M_\odot$. You may find it useful to write $g = GM_\oplus R_\oplus^{-2}$ where $M_\oplus$ and $R_\oplus$ are the mass and radius of the Earth. Their values are given on the equation sheet. Work out the answer to 4 significant figures. You may express your results for Ω in $G = c = 1$ units or MKS units.

Appendix B Curvature Quantities

Schwarzschild Geometry

- **Metric (Schwarzschild coordinates):**

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

- **Christoffel Symbols:**

$\Gamma^t_{tr} = (M/r^2)(1 - 2M/r)^{-1}$

$\Gamma^r_{tt} = (M/r^2)(1 - 2M/r)$

$\Gamma^r_{rr} = -(M/r^2)(1 - 2M/r)^{-1}$

$\Gamma^r_{\theta\theta} = -(r - 2M)$

$\Gamma^r_{\phi\phi} = -(r - 2M)\sin^2\theta$

$\Gamma^\theta_{r\theta} = 1/r$

$\Gamma^\theta_{\phi\phi} = -\cos\theta\,\sin\theta$

$\Gamma^\phi_{r\phi} = 1/r$

$\Gamma^\phi_{\theta\phi} = \cot\theta$

3) Observers 4-velocity $u^\mu = (u^t, 0, 0, \Omega u^t)$

where $u^\phi = \frac{d\phi}{d\tau} = \frac{d\phi}{dt}\frac{dt}{d\tau} = \Omega u^t$

so $u^\mu = u^t(1, 0, 0, \Omega)$

$\underset{\sim}{u} \cdot \underset{\sim}{u} = -1 = (u^t)^2\left(-\left(1 - \frac{2M}{R}\right) + R^2\Omega^2\right)$

so $u^t = \left(1 - \frac{2M}{R} - R^2\Omega^2\right)^{-1/2}$

$\Omega^2 = \frac{M}{R^3}$ for freely falling case

$a^\mu = u^\nu \nabla_\nu u^\mu$

$= \frac{du^\mu}{d\tau} + \Gamma^\mu_{\lambda\kappa} u^\lambda u^\kappa$

$= \Gamma^\mu_{\lambda\kappa} u^\lambda u^\kappa$

By inspection, only non zero comp. is $a^r = \Gamma^r_{tt}(u^t)^2 + \Gamma^r_{\phi\phi}(u^\phi)^2$

$a^r = \left[\frac{M}{R^2}\left(1 - \frac{2M}{R}\right) + \Omega^2(-R + 2M)\right]\left(1 - \frac{2M}{R} - \Omega^2 R^2\right)^{-1}$

$= \frac{\left(1 - \frac{2M}{R}\right)\left(\frac{M}{R^2} - \Omega^2 R\right)}{\left(1 - \frac{2M}{R} - \Omega^2 R^2\right)} = 0$ for $\Omega^2 = \frac{M}{R^3}$ ✓

$$\underset{\sim}{a} \cdot \underset{\sim}{a} = \left(1-\frac{2M}{R}\right)^{-1}(a^r)^2 = \frac{\left(1-\frac{2M}{R}\right)\left(\frac{M}{R}-\Omega^2R^2\right)^2}{1-\frac{2M}{R}-\Omega^2R^2}\ \frac{1}{R^2}$$

so set $\quad g^2R^2 = \dfrac{\left(1-\frac{2M}{R}\right)\left(\frac{M}{R}-\Omega^2R^2\right)^2}{1-\frac{2M}{R}-\Omega^2R^2}$

$$\alpha = \frac{g^2R^2}{1-\frac{2M}{R}} \qquad u = \frac{M}{R} \qquad \nu = \Omega^2R^2$$

$$\alpha = \frac{(u-\nu)^2}{1-2u-\nu}$$ solve for ν

$$\alpha - 2u\alpha - \nu\alpha = u^2 - 2u\nu + \nu^2$$

$$\nu^2 - 2\nu\left(u-\frac{\alpha}{2}\right) + u^2 + 2u\alpha - \alpha = 0$$

$$\nu = u - \frac{\alpha}{2} \pm \left(u^2 - \alpha u + \frac{\alpha^2}{4} - u^2 - 2u\alpha\right.$$

15. [C] A spaceship whose mission is to study the environment around black holes is hovering at a Schwarzschild coordinate radius R outside a spherical black hole of mass M. To escape back to infinity, crew must eject part of the rest mass of the ship to propel the remaining fraction to escape velocity. What is the largest fraction f of the rest mass that can escape to infinity? What happens to this fraction as R approaches $2M$?

12.5 We can begin with radial equation, $\ell = 0$

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{2M}{r} - C$$

C is const. related to e or $\mathcal{E}$

since $\frac{dr}{d\tau} = 0$ at $r = 10M$, $C = \frac{1}{5}$

so $$\frac{dr}{d\tau} = -\left(\frac{2M}{r} - \frac{1}{5}\right)^{1/2}$$

or $$\tau = \int_0^{10M} \frac{dr}{\left(\frac{2M}{r} - \frac{1}{5}\right)^{1/2}} = 2M \int_0^5 \frac{u^{1/2}\,du}{\left(1 - u/5\right)^{1/2}}$$

$$= 2M \times \frac{5\sqrt{5}\pi}{2} = 5\sqrt{5}\pi M$$

so $$u^t = \left(1-\frac{2M}{R}\right)^{-1}\left(1-\frac{2M}{R}+\left(\frac{m_{esc}}{m_{ej}}\right)^2\frac{2M}{R}\right)^{1/2}$$

so $$\left(1-\frac{2M}{R}\right)^{-1/2} = \frac{m_{esc}}{m}\left(1-\frac{2M}{R}\right)^{-1} + \frac{m_{ej}}{m}\left(1-\frac{2M}{R}\right)^{-1}\left(\quad\right)^{1/2}$$

or $$\left(1-\frac{2M}{R}\right)^{1/2} = \frac{m_{esc}}{m} + \left(\left(1-\frac{2M}{R}\right)\left(\frac{m_{ej}}{m}\right)^2+\left(\frac{m_{esc}}{m}\right)^2\frac{2M}{R}\right)^{1/2}$$

solve for $\frac{m_{esc}}{m} \equiv r_{esc}$ $\quad \frac{m_{ej}}{m} = r_{ej}$

$$r_{esc}^2 - 2r_{esc}\left(1-\frac{2M}{R}\right)^{1/2} + 1 - \frac{2M}{R} = \left(1-\frac{2M}{R}\right)r_{ej}^2 + \frac{2M}{R}r_{esc}^2$$

$$\left(1-\frac{2M}{R}\right)r_{esc}^2 - 2r_{esc}\left(1-\frac{2M}{R}\right)^{1/2} + \left(1-\frac{2M}{R}\right)\left(1-r_{ej}^2\right) = 0$$

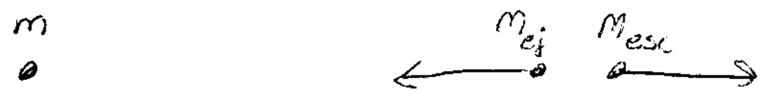
$$r_{esc}^2 - 2\left(1-\frac{2M}{R}\right)^{-1/2}r_{esc} + 1 - r_{ej}^2 = 0$$

$$r_{esc} = \left(1-\frac{2M}{R}\right)^{-1/2} \pm \left(\left(1-\frac{2M}{R}\right)^{-1} - 1 + r_{ej}^2\right)$$

r_{esc} must be lesthan 1 so pick minus sign.

then choose $r_{ej}^2 = 0$ to maximize r_{esc} $\quad r_{esc} = \dfrac{1-\left(2M/R\right)^{1/2}}{\left(1-2M/R\right)^{1/2}}$

12.15



$$p_i^\alpha = m\left(\left(1-\frac{2M}{R}\right)^{-1/2}, 0, 0, 0\right)$$

$$p_{esc}^\alpha = m_{esc}\left(\left(1-\frac{2M}{R}\right)^{-1}, \left(\frac{2M}{R}\right)^{1/2}, 0, 0\right)$$ time-reversal of radial plunge

$$p_{ej}^\alpha = m_{ej}\left(u^t, u^r, 0, 0\right)$$

$$\left(1-\frac{2M}{R}\right)(u^t)^2 - \left(1-\frac{2M}{R}\right)^{-1}(u^r)^2 = 1 \qquad \underset{\sim}{u}\cdot\underset{\sim}{u} = -1$$

$$m\left(1-\frac{2M}{R}\right)^{-1/2} = m_{esc}\left(1-\frac{2M}{R}\right)^{-1} + m_{ej}\, u^t \qquad \text{cons of } p^t$$

$$0 = m\left(\frac{2M}{R}\right)^{1/2} + m_{ej}\, u^r$$

find $\frac{m_{esc}}{m}$

$$\frac{m_{esc}}{m}\left(\frac{2M}{R}\right)^{1/2} = -\frac{m_{ej}}{m}\, u^r$$

$$(u^t)^2 = \left(1-\frac{2M}{R}\right)^{-2}\left(1-\frac{2M}{R} + (u^r)^2\right)$$