Problem Set 6

12.14 The equation $\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r)$

$$V_{eff}(r) = -\frac{M}{r} + \frac{\ell^2}{2r^2}\left(1 - \frac{2M}{r}\right)$$

works in EF or Schwarzschild (same r, same τ!)
+ is therefore valid for $r < 2M$

For $r < 2M$ coeff. of ℓ^2 is negative + so increasing ℓ^2 makes $\left(\frac{dr}{d\tau}\right)^2$ ~~bigger~~ or $-\frac{dr}{d\tau}$ bigger
i.e. only makes descent to $r=0$ faster!

$\Rightarrow$ Set $\ell = 0$. Then $\frac{dr}{d\tau} = -\left(2\mathcal{E} + \frac{2M}{r}\right)^{1/2}$

In the case where $\frac{dr}{d\tau} \to 0$ as $r \to \infty$, $\mathcal{E} = 0$ + we get Eq. 9.37

Here, we want $\frac{dr}{d\tau} = 0$ for $r = 2M$. $\Rightarrow$ $\mathcal{E} = -1/2$

and $\frac{dr}{d\tau} = -\left(\frac{2M}{r} - 1\right)^{1/2}$ $\quad \tau = \int_0^{2M} \frac{dr}{\left(\frac{2M}{r} - 1\right)^{1/2}}$

$\tau = 2M \int_0^1 \frac{u^{1/2}\,du}{(1-u)^{1/2}}$ $\quad u = \sin^2 x \quad du = 2\sin x \cos x\, dx$

$= 2M \int_0^{\pi/2} \frac{\sin x \, 2\sin x \cos x\, dx}{\cos x} = 2M \int_0^{\pi/2} 1 - \cos 2x \, dx = \pi M$

12.4 $$ds^2 = -\left(1-\frac{M}{r}\right)^2 dt^2 + \left(1-\frac{M}{r}\right)^{-2} dr^2 + r^2 d\Omega^2$$

write $t = v - f(r)$ $\quad dt = dv - f'dr$

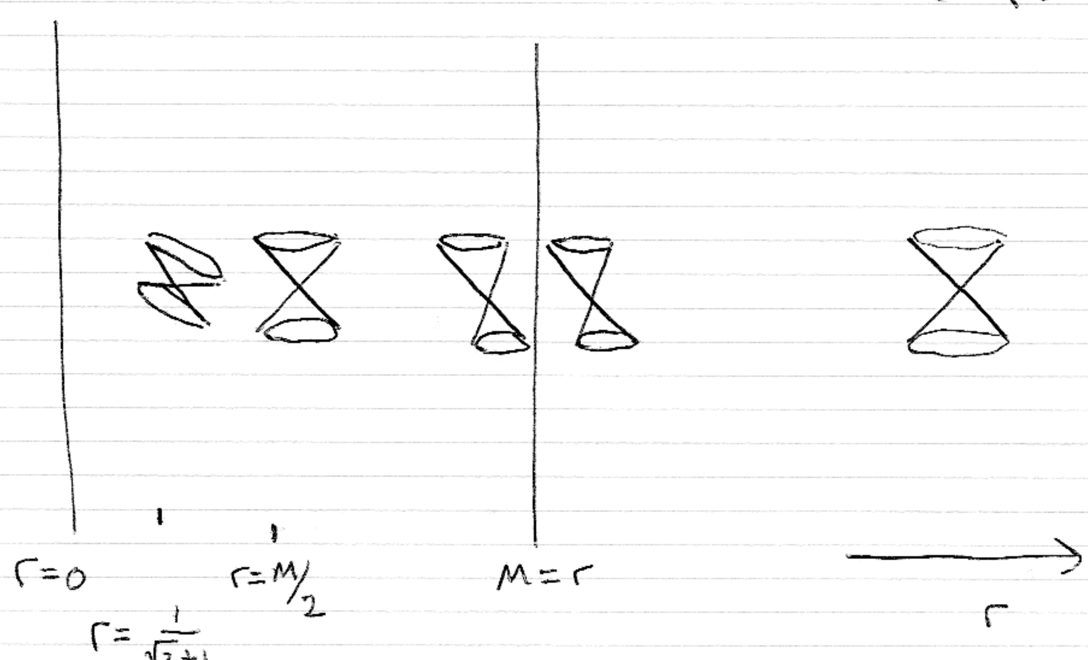
$$ds^2 = -\left(1-\frac{M}{r}\right)^2 dv^2 + 2\left(1-\frac{M}{r}\right)^2 f' dv\,dr - \left(\left(1-\frac{M}{r}\right)^2 f'^2 - \left(1-\frac{M}{r}\right)^{-2}\right) dr^2 + r^2 d\Omega^2$$

so $$f' = \left(1-\frac{M}{r}\right)^{-2}$$

$$\Rightarrow \quad ds^2 = -\left(1-\frac{M}{r}\right)^2 dv^2 + 2\,dv\,dr + r^2 d\Omega^2$$

lightcone $\quad dv = 0 \qquad \frac{dv}{dr} = \frac{2}{\left(1-\frac{M}{r}\right)^2}$

$\tilde{t} = v - r$ so $$\frac{d\tilde{t}}{dr} = \frac{dv}{dr} - 1 = \begin{cases} -1 \\ \frac{2}{\left(1-\frac{M}{r}\right)^2} - 1 \end{cases}$$



particles can move from $r=0$ to $r=M$ but not cross to $r>M$. Particles (and light) can cross from $r>M$ to $r<M$.