Problem Set 9

18.11 a) $a(\eta) = \dfrac{\Omega}{2H_0(\Omega-1)^{3/2}}(1-\cos\eta)$

$$t(\eta) = \frac{\Omega}{2H_0(\Omega-1)^{3/2}}(\eta - \sin\eta)$$

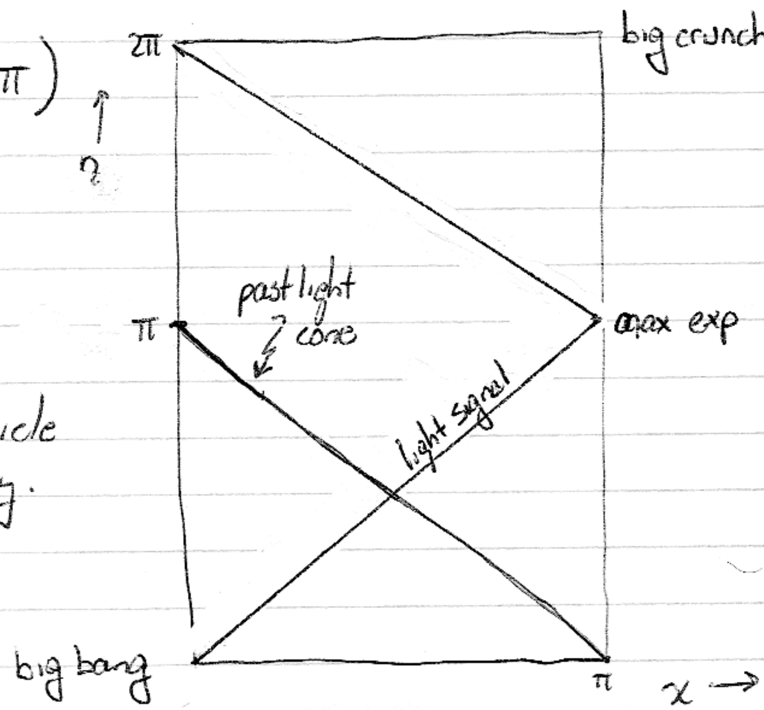
find $\dfrac{dt}{d\eta} = \dfrac{\Omega}{2H_0(\Omega-1)^{3/2}}(1-\cos\eta) = a(\eta)$

$$\Rightarrow \quad ds^2 = -dt^2 + a^2(t)\left(d\chi^2 + \sin^2\chi \, d\Omega^2\right)$$

$$= a^2(\eta)\left(-dt^2 + d\chi^2 + \sin^2\chi \, d\Omega^2\right)$$

b) $a_{max} = a(\eta = \pi) = \dfrac{\Omega}{H_0(\Omega-1)^{3/2}}$

$a_{crunch} = 0 = a(\eta = 2\pi)$



c) YES - light from $\chi = 0$ reaches $\chi = \pi$ in half age of U

d) A light signal does so a particle with $v \to c$ can just barely.

18.20 $$\left(\frac{1}{H_0}\frac{da}{dt}\right)^2 = \Omega_m a^{-1} + \Omega_V a^2 = \Omega_m a^{-1} + (1-\Omega_m) a^2$$

separation of variables

$$\int_0^a \frac{da'}{\left(\Omega_m a'^{-1} + (1-\Omega_m)a'^2\right)^{1/2}} = H_0 t$$

so $$H_0 t = \frac{1}{\Omega_m^{1/2}} \int \frac{a'^{1/2}\, da'}{\left(1 + \frac{1-\Omega_m}{\Omega_m} a'^3\right)^{1/2}}$$

let $$\tan^2 x = \frac{1-\Omega_m}{\Omega_m} a'^3 \qquad \left(1 + \frac{1-\Omega_m}{\Omega_m} a^3\right)^{1/2} = \frac{1}{\cos x}$$

$$\frac{dx}{\cos^2 x} = \left(\frac{1-\Omega_m}{\Omega_m}\right)^{1/2} \frac{3}{2} a'^{1/2} da' \quad \text{or} \quad \frac{a'^{1/2} da'}{\Omega_m^{1/2}} = \frac{2}{3} \frac{1}{(1-\Omega_m)^{1/2}} \frac{dx}{\cos^2 x}$$

$$H_0 t = \frac{2}{3}\frac{1}{(1-\Omega_m)^{1/2}} \int \frac{dx}{\cos x}$$

$$= \frac{2}{3}\frac{1}{(1-\Omega_m)^{1/2}} \ln\left(\frac{1}{\cos x} + \tan x\right)$$

$$H_0 t = \frac{2}{3}\frac{1}{(1-\Omega_m)^{1/2}} \ln\left(\left(1 + \frac{1-\Omega_m}{\Omega_m} a^3\right)^{1/2} + \left(\frac{1-\Omega_m}{\Omega_m} a^3\right)^{1/2}\right)$$

Gives t as function of a

Can use $\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3p)$ $\qquad \rho_{crit} = \frac{3H_0^2}{8\pi}$

$$= -\frac{H_0^2}{2}\left(\frac{\rho + 3p}{\rho_{crit}}\right)$$

$$= -\frac{H_0^2}{2}\left(\Omega_m + \Omega_v - 3\Omega_v\right) \qquad p_v = -\rho$$

$$= -\frac{H_0^2}{2}\left(\Omega_m - 2\Omega_v\right) \qquad \text{want } \Omega_v > \frac{\Omega_m}{2}$$

but $\Omega_v = 1 - \Omega_m$ so $\Omega_v > \frac{1-\Omega_v}{2}$

$$\Omega_v > \frac{1}{3}$$

Alt. $\frac{2}{H_0^2}\frac{da}{dt}\frac{d^2a}{dt^2} = -\frac{\Omega_m}{a^2}\frac{da}{dt} + 2\Omega_v \frac{da}{dt} a$

$$\ddot{a} = -\frac{2}{H_0^2}\left(\frac{\Omega_m}{a^2} - 2\Omega_v a\right) > 0 \text{ for } a=1 \text{ for } \Omega_v > \frac{\Omega_m}{2}$$

$$t_0 = \frac{2}{3H_0}\frac{1}{\Omega_v^{1/2}} \ln\left(\left(1+\frac{1-\Omega_m}{\Omega_m}\right)^{1/2} + \left(\frac{1-\Omega_m}{\Omega_m}\right)^{1/2}\right)$$

$$= \frac{2}{3H_0}\frac{1}{\Omega_v^{1/2}} \ln\left(\left(1+(1-\Omega_m)^{1/2}\right)/\Omega_m^{1/2}\right)$$

$$= \frac{2}{3H_0}\frac{1}{\Omega_v^{1/2}} \ln\left(\frac{1+\Omega_v^{1/2}}{(1-\Omega_v)^{1/2}}\right)$$

for $\Omega_v \to 0$ we get $\frac{2}{3H_0}\frac{1}{\Omega_v^{1/2}} \ln\left(1+\Omega_v^{1/2}\right) \simeq \frac{2}{3H_0}$