QUEEN'S UNIVERSITY
DEPARTMENT OF PHYSICS
PHYS 414, FACULTY OF ARTS AND SCIENCE
PHYS 823, SCHOOL of GRADUATE STUDIES
FINAL EXAMINATION

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INSTRUCTIONS:

The exam is three hours in length. You will be given an equation sheet and may use a calculator. Attempt all four question and write the your answers in the booklets provided. **Each problem is worth 25 points.**
Note that proctors cannot answer questions related to the exam.

GOOD LUCK!

1. Any vector $\mathbf{w}$ satisfies the following equation:

$$(\nabla_\delta \nabla_\gamma - \nabla_\gamma \nabla_\delta)\, w_\beta = w^\alpha R_{\alpha\beta\gamma\delta} \tag{1}$$

where $R_{\beta\alpha\gamma\delta}$ is the Reimann curvature tensor. Consider a Killing vector ξ^α. In addition to Equation 1, ξ^α also satisfies the Killing equation

$$\nabla_\gamma \xi_\beta + \nabla_\beta \xi_\gamma = 0 \tag{2}$$

(a) Show that ξ^α satisfies the following two equations.

$$\nabla_\gamma \xi^\gamma = 0 \tag{3}$$

and

$$\nabla_\alpha \nabla^\alpha \xi_\gamma = -R_{\gamma\alpha} \xi^\alpha \tag{4}$$

(b) Define the rank-2 tensor $F_{\alpha\beta} \equiv \nabla_\alpha \xi_\beta - \nabla_\beta \xi_\alpha$. Show that F satisfies the equation

$$\nabla_\alpha F_{\beta\gamma} + \nabla_\gamma F_{\alpha\beta} + \nabla_\beta F_{\gamma\alpha} = 0 \tag{5}$$

You will find useful formulae on the equation sheet.

2. Consider a spatially flat, FLRW cosmology filled with nonrelativistic matter ($p = 0$) and vacuum energy ($p = -\rho$). That is, the total energy density is given by $\rho = \rho_m + \rho_v$.

   (a) Write out the Friedmann equation for this cosmology. You may find useful formulae and definitions on the equation sheet.

   (b) Work out an expression for the age of the Universe in terms of the present value of the Hubble parameter, H_0 and the density parameter for matter, $\Omega_{m,0} = \rho_{m,0}/\rho_{\text{crit}}$. (The subscript "0" indicates a quantity calculated at the present epoch.) Evaluate the age for a Universe in which $H_0 = 70 \text{km s}^{-1}\,\text{Mpc}^{-1}$ ratio of vacuum energy to ordinary non-relativistic matter at present is 2.3. Compare this age with the age of a flat FLRW cosmology with only non-relativistic matter the same Hubble constant.

   (c) Consider an object of proper size D. Work out the expression for the angular size of the object as a function of redshift. Your expression will involve an integral which you are not expected to do. However, describe, qualitatively, how the angular size of the object changes as you change the relative density in matter and vacuum energy.

   Useful formulae:

$$\int \frac{x^{1/2}dx}{(1+bx^3)^{1/2}} = \frac{2}{3b^{1/2}} \sinh^{-1}(b^{1/2}x^{3/2}) \tag{6}$$

$$\sinh^{-1}(y) = \ln\left(y + \sqrt{y^2+1}\right) \tag{7}$$

$$H_0^{-1} = 1.40 \times 10^{10}\,\text{yr} \quad \text{for} \quad H_0 = 70\,\text{km}\,s^{-1}\,\text{Mpc}^{-1} \tag{8}$$

3. Astronauts wish to orbit the supermassive black hole at the center of the galaxy. Assume the metric is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{9}$$

They decide to follow a circular orbit with $r = R$, $\theta = \pi/2$ and angular velocity $d\phi/dt = \Omega$ where Ω is a constant. Now if they choose the freely orbiting value for Ω, they will experience weightlessness and their bones and muscles will atrophy. They decide instead to pick a non-freely falling orbit.

(a) Calculate the four-acceleration, $a^\mu = u^\nu \nabla_\nu u^\mu$ and its magnitude $a = (a^\mu a_\mu)^{1/2}$ as a function of Ω, R and M. For what value of Ω does $a = 0$?

(b) Suppose the astronauts wish to experience an Earth-like "force". Set $a = g$ and solve for Ω. Note that there are two solutions.

(c) Find the two values of Ω for $R = 10M$, $a = g = 9.8\,\mathrm{m\,s^{-2}}$, and $M = 10^9 M_\odot$. You may find it useful to write $g = GM_\oplus R_\oplus^{-2}$ where $M_\oplus$ and $R_\oplus$ are the mass and radius of the Earth. Their values are given on the equation sheet. Work out the answer to 4 significant figures. You may express your results for Ω in $G = c = 1$ units or MKS units.

4. Consider the metric

$$ds^2 = -(1 - Ar)\,dt^2 + (1 - Ar)^{-1}\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{10}$$

(a) Draw the spacetime diagram for this metric in the $r - t$ plane showing the lightcone structure.

(b) Consider a massive particle that starts at rest at $r = 0$. Find r as a function of the particle's proper time τ. How long, according to the particles own clocks, does it take to reach $r = A^{-1}$.

(c) Find r as a function of coordinate t and comment on the coordinate time t as the particle in part (c) approaches $r = A^{-1}$.

(d) Show that the coordinate singularity at $r = A^{-1}$ can be removed by a transformation of the form

$$t = v - f(r) \tag{11}$$

Find $f(r)$ and write out the metric in terms of v and r. Draw the spacetime diagram showing the lightcone structure in the $v - r$ plane.

Useful formulae

$$\int \frac{dx}{x^{1/2}(1 - ax)} = \frac{2}{a^{1/2}}\text{arctanh}\left((ax)^{1/2}\right) \tag{12}$$

$$\int \frac{dx}{1 - ax^2} = \frac{1}{a^{1/2}}\text{arctanh}\left(a^{1/2}x\right) \tag{13}$$