References
Feynman Lectures III
Chapt 19
Classical Dynamics by
Thornton + Marion

Calculus of Variations

Determine $y(x)$ such that

$$J \equiv \int_{x_1}^{x_2} f(y(x), y'(x); x)\, dx$$

is an extremum

Write $y(\alpha, x) = y(0, x) + \alpha\, \eta(x)$

as most general path. $y(0,x)$ is desired solution. NB $\eta(x_1) = \eta(x_2) = 0$

for extremum, want

$$\frac{\partial J}{\partial \alpha} = \frac{\partial}{\partial \alpha} \int_{x_1}^{x_2} f(y, y'; x)\, dx = 0$$

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} \frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'} \frac{\partial y'}{\partial \alpha} \right) dx$$

$$\frac{\partial y}{\partial \alpha} = \eta(x)$$

$$\frac{\partial y'}{\partial \alpha} = \eta'(x)$$

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left(\eta \frac{\partial f}{\partial y} + \frac{\partial f}{\partial y'} \frac{d\eta}{dx} \right) dx$$

$$= \int_{x_1}^{x_2} \eta \frac{\partial f}{\partial y} - \left(\frac{d}{dx} \frac{\partial f}{\partial y} \right) \eta \; dx$$

$$+ \frac{\partial f}{\partial y'} \eta(x) \Big|_{x_1}^{x_2}$$

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} \right) \eta(x)\, dx$$

$\eta(x)$ is arbitrary + so is $\frac{\partial J}{\partial \alpha} = 0 \quad \forall \eta$
we must have

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

More common notation

$$\delta J = \frac{\partial J}{\partial \alpha} d\alpha$$

$$= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} \right) \frac{\partial y}{\partial \alpha} d\alpha \, dx$$

$$= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} \right) \delta y \, dx$$

$$\delta J = \int_{x_1}^{x_2} \delta f \, dx = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial y'} \delta (y') \right) d$$

$$\delta y' = \delta \frac{dy}{dx} = \frac{d}{dx} \delta y$$ then I by P

Classical NR mechanics

$$L = T - U$$

T = kinetic $\quad$ U = potential

$= T(\dot{x}_i)$ $\quad$ $= U(x_i)$ $\quad$ $\overset{x}{i} = \overset{}{1}, \overset{y}{2}, \overset{z}{3}$

or more general

$$\delta \int_{t_1}^{t_2} L(x, \dot{x}; t) = 0$$

ignore

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0$$

for 1D motion $\quad T = \frac{1}{2} m \dot{x}_i^2$

$$\frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i$$

$L = \frac{1}{2} m \dot{x}_i^2 - U(x$

so we have

$$-\frac{\partial U}{\partial x_i} - m\frac{d}{dt}\dot{x}_i = 0$$

or $\quad m\ddot{x}_i + \frac{\partial U}{\partial x_i} = 0 \qquad m\vec{a} = -\vec{\nabla}U$

e.g. $\quad U = mgy$

$$m\ddot{x} = 0 \qquad m\ddot{y} = -mg \quad \text{or} \quad \ddot{y} = -g$$

Suppose we want equations in polar coordinates. $\quad x = r\cos\theta \quad y = r\sin\theta$

$$T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2$$

$$U = +mgr\sin\theta$$

$$\frac{\partial L}{\partial \dot r} = m\dot r \qquad \frac{\partial L}{\partial \dot\theta} = mr^2\dot\theta$$

$$\frac{\partial L}{\partial r} = -mg\sin\theta + mr\dot\theta^2 \qquad \frac{\partial L}{\partial \theta} = -mgr\cos\theta$$

so $$\frac{d}{dt}\frac{\partial L}{\partial \dot r} = m\ddot r = \frac{\partial L}{\partial r} = -mg\sin\theta + mr\dot\theta^2$$

or $$\boxed{r\dot\theta^2 - g\sin\theta - \ddot r = 0}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} = 2mr\dot r\dot\theta + mr^2\ddot\theta$$

$$= \frac{\partial L}{\partial \theta} = mgr\cos\theta$$

$$\boxed{r\ddot\theta + 2\dot r\dot\theta = -g\cos\theta}$$

Consider particle orbit in central force potential $U = U(r)$

Orbit is in plane so let $\theta = 0$

$$T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 - U(r)$$

$$\frac{\partial T}{\partial \dot{r}} = m\dot{r} \qquad \frac{\partial T}{\partial \dot{\phi}} = mr^2\dot{\phi}$$

+ we have $\quad m\ddot{r} + \dfrac{\partial U}{\partial r} = 0$

and $\dfrac{d}{dt}\left(mr^2\dot{\phi}\right) = 0 \Rightarrow mr^2\dot{\phi} = \text{const!}$

Conserved quantity $-$ $\boxed{L = mr^2\dot{\phi}}$

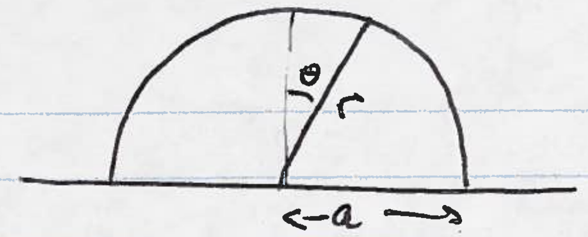
Let $\quad L = L(q_i, \dot{q}_i)$ generalized coordinates

if L is independent of a particular q_j

i.e. $\quad \dfrac{\partial L}{\partial q_j} = 0 \quad$ then $\quad \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}_j} = 0$

and $\quad \dfrac{\partial L}{\partial \dot{q}_j} = $ const of motion.

Constraints



generalized coordinates r, θ

constraint $r - a = 0$

$$T = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2)$$

$$U = mgr\cos\theta$$

$$L = T - U = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\cos\theta$$

Constraint of form $f(q_i, \dot{q}_i) = 0$

for each constraint, introduce of Lagrange multiplier

$$L \rightarrow L = T - U + \lambda f = 0$$

Here, we would write

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\cos\theta + \lambda(r-a)$$

equations

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - mg\cos\theta + \lambda$$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

so $$r\dot{\theta}^2 - g\cos\theta + \frac{\lambda}{m} - \ddot{r} = 0$$

$$\frac{\partial L}{\partial \theta} = mgr\sin\theta \qquad \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

$$gr\sin\theta - r^2\ddot{\theta} - 2\dot{r}\dot{\theta} = 0$$

$$\frac{\partial L}{\partial \lambda} = r - a = 0 \qquad \dot{r} = 0 \qquad \ddot{r} = 0$$

so we have

$$a\dot{\theta}^2 - g\cos\theta + \lambda = 0$$

$$\dot{\theta}\left\{ \cancel{a} g\sin\theta - a^2\ddot{\theta} = 0 \right\}$$

+ integrate $\quad -\dfrac{a\dot{\theta}^2}{2} - g\cos\theta = \text{const}$

at $\theta = 0$ $\quad \dot{\theta}^2 = 0$ so $\quad$ const $= -g$

$$\frac{\dot{\theta}^2}{2} = -\frac{g}{a}\cos\theta + \frac{g}{a}$$

and $\quad \dot{\theta}^2 = \dfrac{g}{a}\cos\theta - \dfrac{\lambda}{a}$

$$\Rightarrow \quad \frac{\lambda}{m} = g\cos\theta - \dot{\theta}^2$$

$$= g\cos\theta + 2g\cos\theta - 2g$$

$$= g(3\cos\theta - 2)$$

$$= 0 \text{ for } \cos\theta = 2/3$$