

Physics 414/823 Midterm; October 24, 2007

1. A certain spacetime is described by the metric

$$ds^2 = - (1 - H^2 r^2) dt^2 + (1 - H^2 r^2)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (1)$$

(a) Describe the lightcone structure in the $r - t$ plane using both equations and a spacetime diagram. Think carefully about the lightcone structure for $r > H^{-1}$ vis-a-vis $r < H^{-1}$.

(b) Construct the embedding diagram for this spacetime. The following steps will guide you through the process:

i. Argue that it is sufficient to consider the two-dimensional slice

$$d\Sigma^2 = (1 - r^2 H^2)^{-1} dr^2 + r^2 d\phi^2 \quad (2)$$

ii. Pick one of the three common forms for the three-dimensional flat space line element:

$$ds_{3D}^2 = dx^2 + dy^2 + dz^2 \quad (3)$$

$$ds_{3D}^2 = d\rho^2 + \rho^2 d\varphi^2 + dz^2 \quad (4)$$

$$ds_{3D}^2 = dw^2 + w^2 (d\Theta^2 + \sin^2 \Theta d\Phi^2) \quad (5)$$

and find the equations that describe the two-dimensional surface corresponding to Eq. (2). What is the geometry of this surface?

2. Consider the motion of massive particles in a spacetime described by the line element

$$ds^2 = -\left(1 + A^2 r^2\right) dt^2 + \left(1 + A^2 r^2\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right) \quad (6)$$

You need only consider orbits in the equatorial ($\theta = 0$) plane.

- (a) Write down the Killing vectors and conserved quantities.
 (b) Derive the equation of motion for particles in this spacetime. Put your equation in the form

$$\frac{1}{2} \left(\frac{dr}{d\tau}\right)^2 + V_{\text{eff}}(r) = \mathcal{E} \quad (7)$$

where \mathcal{E} is a constant.

- (c) Make a sketch of the effective potential. Note that you will need to treat the case of radial orbits separately from the case of non-radial orbits. Describe the different possibilities for particle orbits. Be as quantitative as you can about the limiting forms of the potential, maxima and minima, etc..
 (d) Consider a particle in a circular orbit with $r = R$. Find the orbital period both in terms of the proper time and the coordinate time. You should be able to express your answer solely in terms of A .
 (e) Now consider a particle that starts at rest at $r = R$. Describe the orbit of the particle in this case. Calculate proper time for the particle to return to its original position.
 (f) Write down an integral expression for the coordinate time for the particle in the previous part to return to its original position. Do not attempt the integral (I don't believe it can be done in closed form.) Assume $AR \ll 1$ and evaluate the integral using Taylor expansions, keeping terms of order $A^2 R^2$ but ignoring higher order terms.
 (g) What first-year physics problem does this remind you of? Discuss.

Useful expressions:

$$\int \frac{x dx}{(1-x^2)^{1/2}} = (1-x^2)^{1/2} \quad (8)$$

$$\int_0^1 \frac{dx}{(1-x^2)^{1/2}} = \frac{\pi}{2} \quad \int_0^1 \frac{x^2 dx}{(1-x^2)^{1/2}} = \frac{\pi}{4} \quad (9)$$

$$(1+x)^\alpha = 1 + \alpha x \quad (10)$$