Midterm

Question 1

1 a) factor $1 - \frac{2M(r)}{r} = \begin{cases} 1 - \frac{2M}{r} & r > R \\ 1 - \frac{2M}{R}\left(\frac{r}{R}\right)^2 & r < R \end{cases}$

since $R > 2M$, this factor is never 0!

i.e. $\frac{2M}{R}\left(\frac{r}{R}\right)^2 < 1$ for $r < R$ and $\frac{2M}{R} < 1$

2

coordinate sing. at $r = 0$ & $\theta = 0, \pi$ are usual ones in spherical, polar coordinates.

b) $\xi^\alpha = (1, 0, 0, 0) \qquad \eta^\alpha = (0, 0, 0, 1)$

$$e \equiv -\underset{\sim}{\xi} \cdot \underset{\sim}{u} = \left(1 - \frac{2M(r)}{r}\right)\frac{dt}{d\lambda}$$

3

$$l = \underset{\sim}{\eta} \cdot \underset{\sim}{u} = r^2 \frac{d\phi}{d\lambda} \qquad \theta = \pi/2$$

c) $\underset{\sim}{u} \cdot \underset{\sim}{u} = 0$

or $\left(1 - \frac{2M(r)}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 - \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2 = 0$

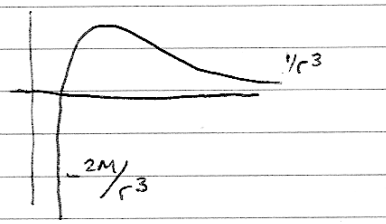
or $e^2 - \left(\frac{dr}{d\lambda}\right)^2 + \frac{l^2}{r^2}\left(1 - \frac{2M}{r}\right) = 0$

or $\frac{e^2}{l^2} \equiv \frac{1}{b^2} = \frac{1}{l^2}\left(\frac{dr}{d\lambda}\right)^2 + W_{eff}$

4 $W_{eff} = \frac{1}{r^2} - \frac{2M(r)}{r^3}$

Schwarz. $\frac{1}{r^2} - \frac{2M}{r^3}$

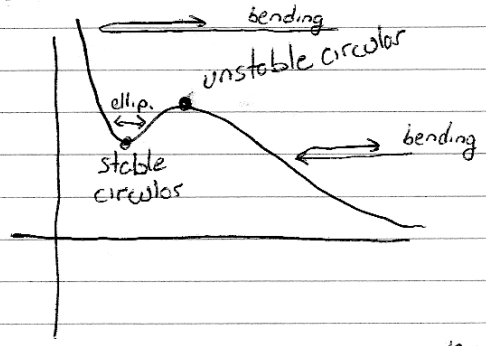
d) $= \begin{cases} \frac{1}{r^2} - \frac{2M}{r^3} & r > R \\ \frac{1}{r^2} - \frac{2M}{R^3} & r < R \end{cases}$



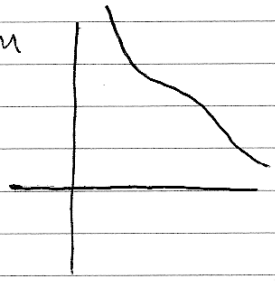
4

Recall $\frac{dW_{eff}}{dr} = -\frac{2}{r^3} + \frac{6M}{r^4}$ in Schwarz. $= 0$ for $r = 3M$

$2M < R < 3M$



$R > 3M$



no horizon where photons fall in + do not escape

e) $$\frac{dt}{dr} = \frac{1}{1 - \frac{2M(r)}{r}}$$

+ since $r < R$ $\quad \frac{2M(r)}{r} = \frac{2M}{R}\frac{r^2}{R^2}$

$$t = \int_0^R \frac{dr}{1 - ar^2} \qquad a = \frac{2M}{R^3}$$

4

$$= \left(\frac{R^3}{2M}\right)^{1/2} \tanh^{-1}\left(\left(\frac{2M}{R^3}\right)^{1/2} R\right) = R\left(\frac{R}{2M}\right)^{1/2} \tanh^{-1}\left(\left(\frac{2M}{R}\right)^{1/2}\right)$$

f) $$\simeq R\left(\frac{R^3}{2M}\right)^{1/2}\left(\left(\frac{2M}{R}\right)^{1/2} + \frac{1}{3}\left(\frac{2M}{R}\right)^{3/2}\right)$$

3 $$\simeq R\left(1 + \frac{1}{3}\frac{2M}{R}\right)$$

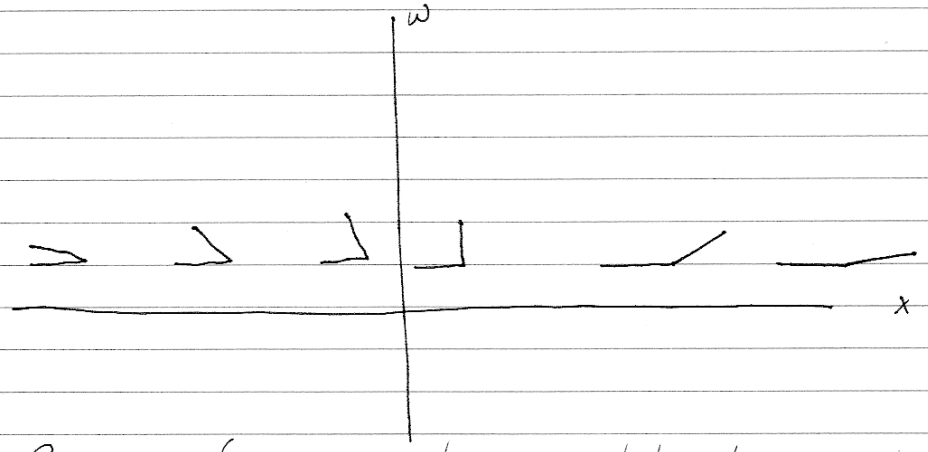
So $$\frac{\delta t}{t} = \frac{1}{3}\frac{2M}{R} = \frac{2 \times 1.5\,\text{km}}{3 \times 7 \times 10^5\,\text{km}} = 1.4 \times 10^{-6}$$

2)

a) $ds^2 = -x\,dw^2 + 2\,dw\,dx$

light cone $\quad 0 = -x\,dw^2 + 2\,dw\,dx$

so $dw = 0$ and $\frac{dw}{dx} = \frac{2}{x}$



Can cross from $x > 0$ to $x < 0$ but not vice versa

b) $ds^2 = -x\,dw^2 + 2\,dw\,dx - \frac{1}{x}dx^2 + \frac{1}{x}dx^2$

$= -x\left(dw - \frac{1}{x}dx\right)^2 + \frac{1}{x}dx^2$

$dv = dw - \frac{1}{x}dx \qquad v = w - \ln x$

$ds^2 = -x\,dv^2 + \frac{1}{x}dx^2$