Problem Set 1

2.7 a) $\quad x = \mu\nu \qquad y = \frac{1}{2}(\mu^2 - \nu^2)$

lines of constant μ $\quad x = \nu\mu \quad y = \frac{1}{2}(\mu^2 - \nu^2)$

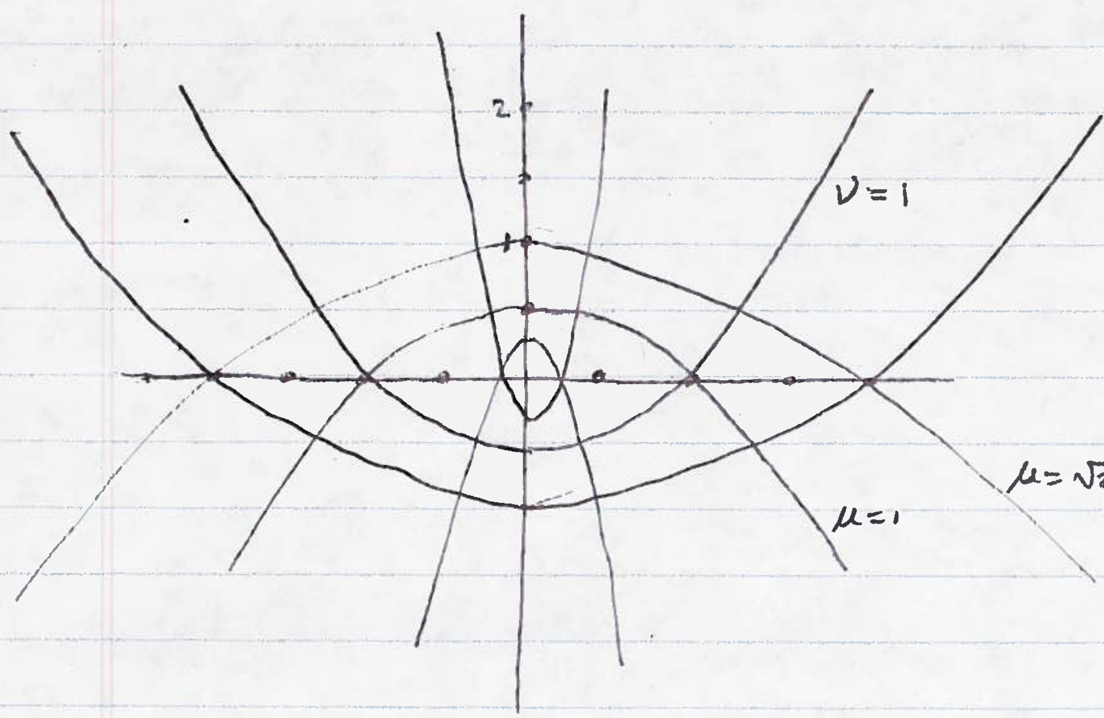
becomes $\quad y = \frac{1}{2}\mu^2 - \frac{1}{2\mu^2}x^2$ i.e. parabola

with $y = \frac{1}{2}\mu^2$ at $x = 0$ and

$y'' = -\frac{1}{\mu^2}$ $\quad$ Note $y = 0$ at $x = \pm\mu$

likewise $\nu = \text{const}$ is a parabola with

$$y = \frac{1}{2\nu^2}x^2 - \frac{\nu^2}{2} \qquad y = 0 \text{ at } x = \pm\nu^2$$



b) $dx = \mu\, d\nu + \nu\, d\mu$

$dy = \mu\, d\mu - \nu\, d\nu$

$$ds^2 = dx^2 + dy^2 = (\mu\, d\nu + \nu\, d\mu)^2 + (\mu\, d\mu - \nu\, d\nu)$$

$$= (\mu^2 + \nu^2)(d\mu^2 + d\nu^2)$$ cross terms cancel!

c) Tangent vector to $\mu = \text{const.}$ $\quad dx = \mu\, d\nu$
$dy = -\nu\, d\nu$

so $\quad dt^\alpha_\mu = (\mu, -\nu)\, d\nu$

for $\nu = \text{const.}$ $\quad dx = \nu\, d\mu \quad dy = \mu\, d\mu$

$\quad dt^\alpha_\nu = (\nu, \mu)\, d\nu$

dot product $\quad d\underset{\sim}{t}_\mu \cdot d\underset{\sim}{t}_\nu = 0$
so vectors are orthogonal.

(there are other ways to show this)

d) $x^2+y^2=r^2$ becomes

$$(\mu\nu)^2 + \frac{1}{4}(\mu^2-\nu^2)^2 = \frac{1}{4}\mu^4 + \frac{1}{4}\nu^4 - \frac{1}{2}\mu^2\nu^2 + \mu^2\nu^2$$

$$= \left(\frac{1}{2}(\mu^2+\nu^2)\right)^2 = r^2$$

or $\mu^2 + \nu^2 = 2r$

e) $C = \oint ds = \oint (\mu^2+\nu^2)^{1/2}(d\mu^2 + d\nu^2)^{1/2}$

Now $\mu d\mu + \nu d\nu = 0$ on circle so

$$d\mu^2 + d\nu^2 = d\mu^2\left(1+\left(\frac{d\nu}{d\mu}\right)^2\right) = d\mu^2\left(1 + \frac{\mu^2}{\nu^2}\right)$$

$$= \frac{d\mu^2}{2r-\mu^2} \times 2r$$

Integrate half circle from $\mu=0$ to $\mu=2r$

$$C = 2 \times 2r \int_0^{2r} \frac{d\mu}{(2r-\mu^2)^{1/2}} = 4r\int_0^1 \frac{dx}{(1-x^2)^{1/2}}$$

$x = \sin\psi \quad (1-x^2)^{1/2} = \cos\psi \quad dx = \cos\psi\, d\psi$

$x = 0$ to $1 \Rightarrow \psi = 0$ to $\pi/2$

$$C = 4r\int_0^{\pi/2} d\psi = 2\pi r \checkmark$$

2.9 $ds^2 = (d\theta^2 + f^2(\theta)\, d\phi^2)$ $f(\theta) = \sin\theta(1 + \varepsilon \sin^2\theta)$

from pole to equator set $d\phi = 0$

$$C_1 = \int_0^{\pi/2} a\, d\theta = \frac{\pi a}{2} \Rightarrow a = \frac{2C_1}{\pi} \qquad C_1 = 9985.16$$

at equator $C_2 = \int_0^{\pi/2} a f(\theta = \pi/2)\, d\phi$

$$= \frac{a(1+\varepsilon)\pi}{2} = 10\,018.75\text{km}$$

$$1 + \varepsilon = \frac{2C_2}{\pi a} = \frac{C_2}{C_1} = 1.00336 \qquad \varepsilon = 0.00336$$

$$a = 6356.75 \text{ km}$$

$$ds^2 = a^2\left(d\theta^2 + f^2(\theta)\, d\phi^2\right)$$

$$dA = a^2\, d\theta\, f(\theta)\, d\phi$$

Area of circular cap out to $\theta = \Theta$

$$A = a^2 \int_0^{\Theta} d\theta\, f(\theta) \int_0^{2\pi} d\phi \qquad f(\theta) = \sin\theta\left(1 + \varepsilon \sin^2\theta\right)$$

$$\sin\theta\, d\theta = -d\cos\theta = du$$

$$1 + \varepsilon \sin^2\theta = 1 + \varepsilon - \varepsilon\cos^2\theta$$

$$= 1 + \varepsilon - \varepsilon u^2$$

$$A = 2\pi a^2 \int_{\cos\Theta}^{1} du\left(1 + \varepsilon - \varepsilon u^2\right)$$

$$= 2\pi a^2 \left\{ (1+\varepsilon)(1 - \cos\Theta) - \frac{\varepsilon}{3}\left(1 - \cos^3\Theta\right) \right\}$$

6 6.13 $$\tau_{AB} = \int_A^B dt\left[1 - \frac{1}{c^2}\left(\frac{v^2}{2} - \phi(\vec{r})\right)\right]$$

y = height above ground $\phi = gy$

a) obs. at ground $v = 0$, $y = 0$

1 then $\tau_{AB} = \int_A^B dt = T$ by def.

b) particle in free-fall

2 $y = v_0 t - \frac{1}{2}gt^2$ $v = v_0 - gt$

max height at $t = v_0/g = T/2$ i.e. $v_0 = gT/2$

max height $= v_0^2/2g = gT^2/8$

so write $y = \frac{gT^2}{2}(u - u^2)$ $v = \frac{gT}{2}(1-2u)$

where $u = t/T$

$$\tau = 2T\int_0^{1/2} du\left[1 - \frac{g^2T^2}{c^2}\left(\frac{1}{8}(1-4u+4u^2) - \frac{1}{2}(u-u^2)\right)\right]$$

$$= 2T\int_0^{1/2} du\left[1 - \frac{g^2T^2}{c^2}\left(\frac{1}{8} - u + u^2\right)\right]$$

$$= T - \frac{2g^2T^3}{c^2}\left(\frac{1}{16} - \frac{1}{8} + \frac{1}{24}\right)$$

$$= T\left(1 + \frac{g^2T^2}{24c^2}\right)$$

c) constant speed case $v = gT/4$ $y = gTt/4$

1st leg $= \frac{gT^2}{4}\frac{t}{T}$

2

$$\tau = 2T\int_0^{1/2} du \left(1 - \frac{1}{c^2}\left(\frac{g^2T^2}{32} - \frac{g^2T^2}{4}u\right)\right)$$

$$= 2T\left(\frac{1}{2} - \frac{g^2T^2}{c^2}\left(\frac{1}{64} - \frac{1}{32}\right)\right)$$

$$= T\left(1 + \frac{g^2T^2}{32c^2}\right)$$

1 free-falling clock has longest time. (b/c in free-falling frame)

8 7.14 $ds^2 = -(1-Ar^2)^2 dt^2 + (1-Ar^2)^2 dr^2 + r^2 d\Omega^2$

2 a) $L_0 = \int_0^R (1-Ar^2)\,dr = R\left(1 - \frac{AR^2}{3}\right)$

2 b) $A = \iint R^2 \sin\theta\, d\theta\, d\phi = 4\pi R^2$

2 c) $V = \int (1-Ar^2)\, r^2 dr \sin\theta\, d\theta\, d\phi$

$= 4\pi \int_0^R (r^2 - Ar^4)\,dr = \frac{4\pi R^3}{3}\left(1 - \frac{3AR^2}{5}\right)$

or $4\pi\left(\frac{R^3}{3} - \frac{AR^5}{5}\right)$

2 d) $V_4 = \int (1-Ar^2)^2 r^2 dr\, d\theta \sin\theta\, d\phi\, dt$

$= 4\pi T \int (r^2 - 2Ar^4 + A^2 r^6)\,dr$

$= \frac{4\pi T R^3}{3}\left(1 - \frac{6AR^2}{5} + \frac{3A^2R^4}{7}\right)$

or $4\pi T\left(\frac{R^3}{3} - \frac{2AR^5}{5} + \frac{A^2R^7}{7}\right)$

Problem Set 3

8.9 $$ds^2 = -X^2 dT^2 + dX^2$$

now we have $$-1 = -X^2\left(\frac{dT}{d\tau}\right)^2 + \left(\frac{dX}{d\tau}\right)^2 \qquad (1)$$

we also have $$L = \left(X^2\left(\frac{dT}{d\sigma}\right)^2 - \left(\frac{dX}{d\sigma}\right)^2\right)^{1/2}$$

+ $\frac{\partial L}{\partial T} = 0$ so $$\frac{\partial L}{\partial\, dT/d\sigma} = \text{const}$$

or $$X^2 \frac{dT}{d\tau} = e \qquad (2)$$

(1) + (2) give $$1 = \frac{e^2}{X^2} - \left(\frac{dX}{d\tau}\right)^2$$

$$\frac{dX}{d\tau} = \pm\left(\frac{e^2}{X^2} - 1\right)^{1/2}$$

we want $X = X(T)$ (not $X = X(\tau)$)

$$\frac{dX}{dT} = \frac{dX}{d\tau}\Big/\frac{dT}{d\tau} = \pm\left(\frac{e^2}{X^2} - 1\right)^{1/2}\Big/ \frac{e}{X^2}$$

or $$\frac{e\,dx}{x^2\left(\frac{e^2}{x^2}-1\right)^{1/2}} = dT$$

$$y = \frac{e}{x} \qquad dy = -\frac{e}{x^2}dx$$

$$\pm\frac{dy}{(y^2-1)^{1/2}} = dT$$

$$y = \cosh u$$
$$y^2-1 = \sinh^2 u$$
$$dy = \sinh u\, du$$

$$\boxed{T = \pm\cosh^{-1}\left(\frac{e}{x}\right) + T_0}$$

8.12 a) On x-axis, $y=0$

$$d = \int_{y=0}^{y=Y} ds = \int_0^Y \frac{dy}{y} = \infty$$ log diverge

b) $g_{xx} = y^{-2}$ $\quad g_{yy} = y^{-2}$ $\quad g^{xx} = g^{yy} = y^2$

$g_{xx,x} = 0$ $\quad g_{xx,y} = -\frac{2}{y^3}$ $\quad g_{yy,y} = -\frac{2}{y^3}$

$$\Gamma^x_{xy} = \frac{1}{2} g^{xx} g_{xx,y} = -\frac{1}{y} = \Gamma^x_{yx}$$

$$\Gamma^y_{xx} = -\frac{1}{2} g^{yy}(-g_{xx,y}) = \frac{1}{y} = -\Gamma^y_{yy}$$

$$\frac{d^2x}{ds^2} = -\Gamma^x_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = 2\frac{1}{y}\frac{dx}{ds}\frac{dy}{ds}$$

$$\frac{d^2y}{ds^2} = -\Gamma^y_{xx}\left(\frac{dx}{ds}\right)^2 - \Gamma^y_{yy}\left(\frac{dy}{ds}\right)^2 = -\frac{1}{y}\left(\frac{dx}{ds}\right)^2 + \frac{1}{y}\left(\frac{dy}{ds}\right)^2$$

c) x - equation integrates

$$\frac{\frac{d^2x}{ds^2}}{\frac{dx}{ds}} = \frac{2\,dy/ds}{y}$$

or $\ln \frac{dx}{ds} = 2 \ln y + c$

$$\frac{dx}{ds} = c y^2$$

we also have $\frac{1}{y^2}\left(\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2\right) = 1$

so $c^2 y^4 + \left(\frac{dy}{ds}\right)^2 = y^2$

$$\frac{dy}{ds} = \pm\left(y^2 - c^2 y^4\right)^{1/2}$$

find $\dfrac{dy}{dx} = \dfrac{dy/ds}{dx/ds}$

$$= \frac{\pm\left(y^2 - c^2y^4\right)^{1/2}}{cy^2}$$

or $$\frac{c\,y\,dy}{\left(1-c^2y^2\right)^{1/2}} = dx$$

or $$\left(1 - c^2y^2\right)^{1/2} = cx$$

$$1 = c^2\left(x^2 + y^2\right) \quad ✓$$