The Wave Equation for transverse waves: summary

→ Sum forces on a small piece of a continuous body (in this case a string)

\[
\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}
\]

WAVE EQUATION (for a stretched string)

The solution for normal modes (called stationary waves, standing waves, or resonances - where \( \omega \) is the same for all points along string)

\[
y(x,t) = f(x) \cos \omega t
\]

evaluating this using BCs for a string fixed both ends

\[f(x) = A \sin \frac{2\pi x}{\lambda_n}
\]

Normal mode frequency

\[\omega_n = \frac{n\pi}{L} \left( \frac{T}{\mu} \right)^{1/2} = n \omega_1
\]

along the way:

\[\omega_n = \frac{n\pi}{L} \left( \frac{T}{\mu} \right)^{1/2} = n \omega_1
\]

for string fixed both ends

\[f(x) = A \sin \frac{2\pi x}{\lambda_n}
\]

Amplitude along the string at any point \( x \)

\[\omega_n = \frac{n\pi}{L} \left( \frac{T}{\mu} \right)^{1/2} = n \omega_1
\]

Normal mode frequency

\[\omega_n = \frac{n\pi}{L} \left( \frac{T}{\mu} \right)^{1/2} = n \omega_1
\]

mass/length

\[\omega_n = \frac{n\pi}{L} \left( \frac{T}{\mu} \right)^{1/2} = n \omega_1
\]

frequency of normal mode or standing wave on string

\[\omega_n = \frac{n\pi}{L} \left( \frac{T}{\mu} \right)^{1/2} = n \omega_1
\]

This holds for BC of \( f(x) = 0 \) at each end of the string

\[\omega_n = \frac{n\pi}{L} \left( \frac{T}{\mu} \right)^{1/2} = n \omega_1
\]

The \( y \) position at any point \( x \) along a string and at time \( t \)

for transverse waves

\[y(x,t) = A_n \sin \frac{2\pi x}{\lambda_n} \cos \omega_n t
\]

WAVE FUNCTION for standing waves (normal modes) of a string!
Week 7 Lecture 3:
no problems for this lecture
Normal Mode solutions to the wave equation – Longitudinal waves

So far we have developed the Wave equation for a transverse wave on a string

Solution for normal modes (wave function) for transverse waves on a string with both ends fixed.

We still have to look at the travelling wave solution, but first we should look at the other type of wave that can propagate in an elastic medium

Longitudinal waves
We will have to start again by developing another wave equation for these kinds of waves. Fortunately the procedure is the same as before, and the result also similar.
The Wave Equation for Longitudinal Waves

→ So far we have only considered transverse waves, where the oscillation of a point is transverse to the wave propagation direction.

→ Now we will consider longitudinal waves, where the oscillation of a point is parallel to the wave propagation direction (sound travels this way).

**Transverse Waves** (reminder-this is what we did before)

→ wave exists all along the string but we are considering only a segment $\Delta x$

→ $\Delta x$ has a transverse displacement $y$

[Diagram of transverse waves with forces and displacement](image)

**Longitudinal Waves: consider a rod**

**Before Wave**

[Diagram of a rod before wave](image)

**In presence of wave**

[Diagram of a rod in presence of wave](image)

→ wave exists all through the rod but we just consider a small segment $\Delta x$

→ in the presence of the wave our segment $\Delta x$ shifts and stretches by a longitudinal displacement $\Delta \eta$
The Wave Equation for Longitudinal Waves in a Rod
(develop using similar procedure to stretched strings)

→ Strike the end of a rod → you create a strain disturbance which moves along the rod

→ Our slice is shifted (due to cumulative strains) and also stretched by $\Delta \eta$
  → experiences forces:
    - $F_1$ (at front of slice)
    - $F_2$ (at back of slice)

→ $\therefore$ average strain across slice $= \frac{\Delta \eta}{\Delta x}$

→ $\therefore$ average stress across slice $= Y \frac{\Delta \eta}{\Delta x} = \tau$

(assuming stress is in the longitudinal direction)
To begin our wave equation determination, we need to sum the forces on our slice (like we did for the string earlier). Here we first look at stresses, then calculate the forces from the stresses. (note \( \eta \) is just the longitudinal displacement or stretch)

**stress at point \( x \):**

\[
\text{stress at } x = Y \frac{\partial \eta}{\partial x}
\]

(partial derivative because here strain is also time dependent)

**stress at point \( x + \Delta x \)**

\[
\text{stress at } x + \Delta x = (\text{stress at } x) + \frac{\partial (\text{stress})}{\partial x} \Delta x
\]

\[
= Y \frac{\partial \eta}{\partial x} + \frac{\partial}{\partial x} \left( Y \frac{\partial \eta}{\partial x} \right) \Delta x
\]

\[
= Y \frac{\partial \eta}{\partial x} + Y \frac{\partial^2 \eta}{\partial x^2} \Delta x
\]
Now we have the stresses at each end, let's get the forces...

\[ F = (\text{cross sectional area})(\text{stress}) \]

\[ \therefore F_1 = \text{force at } x \quad = \alpha Y \frac{\partial \eta}{\partial x} \quad \alpha = \text{cross sectional area} \]

\[ F_2 = \text{force at } x + \Delta x \quad = \alpha Y \frac{\partial \eta}{\partial x} + \alpha Y \frac{\partial^2 \eta}{\partial x^2} \Delta x \]

\[ \sum F \quad \text{gives} \quad F_2 - F_1 = \alpha Y \frac{\partial^2 \eta}{\partial x^2} \Delta x \]

also \[ F_2 - F_1 = ma \quad \text{(on our slice)} \]

\[ m = \rho \alpha \Delta x \quad a = \frac{\partial^2 \eta}{\partial t^2} \]

becomes \[ \alpha Y \frac{\partial^2 \eta}{\partial x^2} \Delta x = (\rho \alpha \Delta x) \left( \frac{\partial^2 \eta}{\partial t^2} \right) \]

\[ \text{cancelling terms gives} \quad \frac{\partial^2 \eta}{\partial x^2} = \frac{\rho}{Y} \frac{\partial^2 \eta}{\partial t^2} \]

\[ \text{WAVE EQUATION for longitudinal waves!} \]

\[ \text{or, if} \quad v = \sqrt{\frac{Y}{\rho}} \]

\[ \frac{\partial^2 \eta}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \eta}{\partial t^2} \]

\[ \text{Note that } \eta \text{ is just the longitudinal displacement, just like } y \text{ was the transverse displacement for transverse waves} \]
Normal Mode Solutions to Longitudinal Wave Equation (similar to transverse - earlier)

→ Normal Mode Solution is of the type

\[ \eta(x, t) = f(x) \cos \omega t \]  \( \text{(1)} \)

→ evaluate \( f(x) \) (put back into wave eqn, same soln as for transverse p.134)

\[ f(x) = A \sin \left( \frac{\omega x}{v} \right) + B \cos \left( \frac{\omega x}{v} \right) \]

→ **Boundary Conditions:** for the string we had both ends clamped. This time we will only clamp one end (we could clamp the other end but we won’t – just to be different than last time).

**BC #1:** fixed end \( f(x)=0 \) at \( x=0 \)  \( \text{BC #2: free end} \) ??? at \( x=L \)

→ **Using BC #1** - same as for transverse waves on a string

\[ f(x) = A \sin \left( \frac{\omega x}{v} \right) \]  \( \text{(2)} \)

→ **What about BC #2?** (this will tell us what \( \omega_n \) can be)

→ At \( x=L \) we have a free end. This implies zero stress at this point.

\[ F = \alpha Y \frac{\partial \eta}{\partial x} = 0 \text{ at } x = L \]

i.e. the very last row of atoms

\[ \text{cross sectional area} \]

so \[ \frac{\partial \eta}{\partial x} = 0 \text{ at } x = L \]

Note: this is the case because there are no stresses in any other directions to give us Poisson’s ratio effects
To use this boundary condition we need to differentiate eqn \( 1 \) but lets put equation \( 2 \) in first

\( 1 + 2 \) \( \eta(x, t) = A \sin \left( \frac{\omega x}{v} \right) \cos \omega t \)

differentiate wrt x \( \frac{\partial \eta}{\partial x} = \frac{A \omega}{v} \cos \left( \frac{\omega x}{v} \right) \cos \omega t \)

apply BC \#2 \( \frac{\partial \eta}{\partial x} = 0 \) at \( x = L \) \( 0 = \frac{A \omega}{v} \cos \left( \frac{\omega L}{v} \right) \cos \omega t \)

must = 0 in order to have \( \frac{\partial \eta}{\partial x} = 0 \) always

\[ \frac{\omega L}{v} \text{ must } = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots \] or \( \frac{\omega L}{v} = \left( n - \frac{1}{2} \right) \pi \) with \( n = 1, 2, \ldots \)

\[ \omega_n = \frac{\left( n - \frac{1}{2} \right) v \pi}{L} \] or with \( v = \sqrt{\frac{Y}{\rho}} \)

\[ \omega_n = \frac{\left( n - \frac{1}{2} \right) \pi}{L} \sqrt{\frac{Y}{\rho}} \]

These are the NM frequencies for a bar with one clamped and one free end, but also same for air channels with one end open

\[
\text{relationship between } \omega_n \text{ and } \omega_1 \text{ when one end free} \\
\omega_n = C \omega_1 \implies \left( n - \frac{1}{2} \right) \pi \sqrt{\frac{Y}{L \rho}} = C \left[ \frac{\pi}{2L} \sqrt{\frac{Y}{\rho}} \right] \\
C = 2 \left( n - \frac{1}{2} \right) = 2n - 1 \\
\text{so } \omega_n = \left( 2n - 1 \right) \omega_1 \\
\omega_2 = 3 \omega_1 \\
\omega_3 = 5 \omega_1 \\
\omega_4 = 7 \omega_1 \]
Finally, we can write $\omega_n$ in terms of $f_n$ (divide by $2\pi$)

\[
f_n = \frac{(n - \frac{1}{2}) }{2L} \sqrt{\frac{Y}{\rho}} \quad \text{or} \quad f_n = (2n - 1)f_1
\]

in Hz
Wave Equations and Functions

**Transverse Waves** (eg. stretched string)

\[
\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}
\]

\[
v = \sqrt{\frac{T}{\mu}}
\]

normal mode solutions (standing waves)

\[
y(x,t) = f(x)\cos \omega t
\]

Boundary Conditions: both ends fixed

\[
y(x,t) = A_n \sin \left( \frac{\omega_n x}{v} \right) \cos \omega_n t
\]

where

\[
\omega_n = \frac{n\pi}{L} \left( \frac{T}{\mu} \right)^{\frac{1}{2}} = n\omega_1
\]

Also (in Hz)

\[
f_n = \frac{\omega}{2\pi} = \frac{n}{2L} \left( \frac{T}{\mu} \right)^{\frac{1}{2}}
\]

**Longitudinal Waves** (eg. vibrating rod)

\[
\frac{\partial^2 \eta}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \eta}{\partial t^2}
\]

\[
v = \sqrt{\frac{Y}{\rho}}
\]

normal mode solutions (standing waves)

\[
\eta(x,t) = f(x)\cos \omega t
\]

Boundary Conditions: one end fixed, one end free

\[
\eta(x,t) = A_n \sin \left( \frac{\omega_n x}{v} \right) \cos \omega_n t
\]

where

\[
\omega_n = \frac{(n-\frac{1}{2})\pi}{L} \left( \frac{Y}{\rho} \right)^{\frac{1}{2}} = (2n-1)\omega_1
\]

Also (in Hz)

\[
f_n = \frac{\omega}{2\pi} = \frac{(n-\frac{1}{2})}{2L} \left( \frac{Y}{\rho} \right)^{\frac{1}{2}}
\]
A Comparison of These 2 Waves
(note these equations are only for specific boundary conditions)

<table>
<thead>
<tr>
<th>Transverse 2 fixed ends</th>
<th>Longitudinal one end fixed</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_n = \frac{n}{2L}\left(\frac{T}{\mu}\right)^{\frac{1}{2}}$</td>
<td>$f_n = \left(\frac{n-\frac{1}{2}}{2L}\right)\left(\frac{Y}{\rho}\right)^{\frac{1}{2}}$</td>
</tr>
</tbody>
</table>

This implies that the string can accommodate an integral number of half sine curves

This implies that the length accommodates an integral number of quarter sine curves

At a snapshot in time the different harmonics look like:

<table>
<thead>
<tr>
<th>Transverse 2 fixed ends</th>
<th>Longitudinal one end fixed</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n=1$</td>
<td>$n=1$</td>
</tr>
<tr>
<td>one half-sin curve</td>
<td>one quarter-sin curve</td>
</tr>
<tr>
<td>$n=2$</td>
<td>$n=2$</td>
</tr>
<tr>
<td>two half-sin curves</td>
<td>three quarter-sin curves</td>
</tr>
<tr>
<td>$n=3$</td>
<td>$n=3$</td>
</tr>
<tr>
<td></td>
<td>five quarter-sin curves</td>
</tr>
</tbody>
</table>

(note: the above is really a longitudinal wave but I have represented it as a transverse one) - see website for longitudinal wave demos
The Wave Equation Revisited

→ General Wave Equation (1D): \[ \frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2} \]

where \( v \) is the wave speed

Can write this for all sorts of situations!

→ stretched string (transverse oscillations): \[ \frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu} \frac{\partial^2 y}{\partial x^2} \quad v = \sqrt{\frac{T}{\mu}} \]

transverse displacement

→ longitudinal waves* in a solid: \[ \frac{\partial^2 \eta}{\partial t^2} = \frac{Y}{\rho} \frac{\partial^2 \eta}{\partial x^2} \quad v = \sqrt{\frac{Y}{\rho}} \]

longitudinal displacement

→ transverse (shear) waves in a solid: \[ \frac{\partial^2 y}{\partial t^2} = \frac{n}{\rho} \frac{\partial^2 y}{\partial x^2} \quad v = \sqrt{\frac{n}{\rho}} \]

→ longitudinal waves in a gas or liquid: \[ \frac{\partial^2 \eta}{\partial t^2} = B \frac{\partial^2 \eta}{\partial x^2} \quad v = \sqrt{\frac{B}{\rho}} \]

*Note: for longitudinal waves in a solid the elastic constant is “\( Y \)” only for a thin rod. When the transverse dimension is large then the correct modulus is for a constrained situation (see constrained modulus earlier).
Normal Mode in 2D and 3D Systems (French pg 181-189)

1D System: BC's at ends determine NM characteristics
2D System: BC's at edges determine NM characteristics
3D System: BC's at surfaces determine NM characteristics
2D and 3D are an extension of 1D!

For transverse wave - fixed both end (or all 4 edges for 2D)

1D

wave equation:
\[ \frac{\partial^2 y}{\partial x^2} = \frac{\mu}{T} \frac{\partial^2 y}{\partial t^2} \]

normal mode solution:
\[ y_n(x,t) = A_n \sin\left(\frac{n\pi x}{L}\right) \cos \omega_n t \]
(nodes [zero amplitude] are points)

normal mode frequencies:
\[ \omega_n = \frac{n\pi}{L} \left(\frac{T}{\mu}\right)^\frac{1}{2} \]

mu = linear density
T = tension

2D

wave equation:
\[ \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\sigma}{s} \frac{\partial^2 z}{\partial t^2} \]

normal mode solution:
\[ z_{n,n_z}(x,y,t) = C_{n,n_z} \sin\left(\frac{n_1\pi x}{L_x}\right) \sin\left(\frac{n_2\pi y}{L_y}\right) \cos \omega_{n,n_z} t \]
(nodes are lines)

normal mode frequencies:
\[ \omega_{n,n_z} = \left(\frac{s}{\sigma}\right)^\frac{1}{2} \left[\left(\frac{n_1\pi}{L_x}\right)^2 + \left(\frac{n_2\pi}{L_y}\right)^2\right]^\frac{1}{2} \]

\[ s = \text{surface tension} \]
\[ \sigma = \text{mass/unit area} \]
\[ L_x, L_y = \text{side lengths of membrane} \]
\[ n_1, n_2 = 1, 2, 3... \]

3D wave equation:
\[ \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial^2 \Psi}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 \Psi}{\partial t^2} \]

eg. in a gas - \( \Psi \) might be a pressure magnitude at any given position and time
\[ v = \sqrt{\frac{B}{\rho}} \]