Noise

- fluctuations in voltage (or current) about zero
- average voltage = 0 \( \langle V \rangle = \int V(t) \, dt = 0 \)
- average \( V^2 \) is not zero \( \langle V^2 \rangle = \int V^2(t) \, dt > 0 \) \( V_{\text{rms}} \equiv \sqrt{\langle V^2 \rangle} \)
  - so, we talk about *rms voltage* for noise
Sources of Intrinsic Noise

- sometimes noise results from inherent fluctuations in the system being measured, for example:
  - twinkling starlight due to atmospheric turbulence randomly refracting a point source of light
  - statistical, physical processes (e.g. thermionic emission of electrons off a hot wire)
  - bad electrical connections at contacts result in vibrations randomly making/breaking surface contact and changing resistance
- there is also external noise – not the subject of this lecture
  - e.g. RF pickup
- for some intrinsic noise sources, we can be quantitative about their intensity...

Thermal or Johnson Noise

- thermal motion of electrons in a resistor gives rise to fluctuating resistance values → producing fluctuating voltages across the resistor
- this additional voltage is independent of whether current is flowing in the resistor or not
- first studied in 1928 by John B. Johnson at Bell Labs
  - Johnson explained his measurements to Harry Nyquist (also Bell Labs) who derived an expression to describe this noise using thermodynamics arguments
- so sometimes called Nyquist noise or Johnson-Nyquist noise
Derivation: Johnson Noise

- A resistor has two terminals and across any pair of terminals is capacitance.
- Energy stored in a capacitor: \( \frac{1}{2}CV^2 \)
- Potential energy in the capacitor determines the kinetic energy of the thermal motion of the charges and vice versa.
  - Characterized by a distribution and temperature.
  - Probability of finding voltage between \( V \) and \( (V+dV) \):
    \[ dP = K_0 e^{-\frac{CV^2}{2k_\text{B}T}} dV \]
    - The usual Boltzmann factor.
  - \( K_0 \) is just a normalization constant.

Johnson Noise Derivation cont’d

- Variable substitution:
  \[ dP = K_0 e^{-\frac{CV^2}{2k_\text{B}T}} dV \]
  \[ x^2 = \frac{CV^2}{2k_\text{B}T}, \quad 2x \, dx = \frac{C}{2k_\text{B}T} dV, \quad dV = \frac{2k_\text{B}T}{C} x \, dx \]
  \[ dV = \frac{2k_\text{B}T}{C} \sqrt{\frac{CV^2}{2k_\text{B}T}} dx, \quad dV = \sqrt{\frac{2k_\text{B}T}{C}} \, dx \]

\[ K_0 \sqrt{\frac{2k_\text{B}T}{C}} \int_{-\infty}^{\infty} e^{-x^2} dx = 1, \]

\[ K_0 = \sqrt{\frac{C}{2\pi k_\text{B}T}}, \quad \text{by setting integrated probability} = 1 \]

We have determined \( K_0 \).
Derivation cont'd

- find mean squared voltage:
  \[ \overline{V^2} = \sqrt{\frac{C}{2\pi k_B T}} \int_{-\infty}^{\infty} V^2 \exp\left(-\frac{CV^2}{2k_B T}\right) dV, \]
  \[ dP = K_0 e^{-\frac{CV^2}{2k_B T}} dV \]

- same variable substitution and the Gaussian integral:
  \[ \overline{V^2} = \sqrt{\frac{C}{2\pi k_B T}} \left(\frac{2k_B T}{C}\right)^{\frac{3}{2}} \int_{-\infty}^{\infty} x^2 e^{-x^2} dx \]

- tells us that:
  \[ \overline{V^2} = \frac{k_B T}{C} \]

makes sense: the equipartition theorem in thermodynamics tells us that on average half the thermal energy goes into potential and half into kinetic

Model Resistor with Voltage Source

- next, consider an intrinsic noise voltage source as part of R:
  - define \( S_V(0) \, df \) as the mean squared voltage of the source \( v_n \) per interval of frequency
  - \( S(f) \) is a power spectral density

\[ d\overline{V_n^2} = S_V(0) df \left(\frac{\omega C}{R^2 + (\omega C)^2}\right)^2 = \frac{S_V(0) df}{1 + (\omega RC)^2} \]

\[ \overline{V_n^2} = \int_0^\infty d\overline{V_n^2} = S_V(0) \int_0^\infty \frac{df}{1 + (\omega RC)^2} \]

\[ \overline{V_n^2} = \frac{S_V(0)}{2\pi RC} \int_0^\infty \frac{dx}{1 + x^2} = \frac{1}{\pi} \int_0^\infty \frac{dx}{1 + x^2} \]

\[ |V_C|^2 = \frac{1}{1 + (\omega RC)^2} \]

mean squared voltage of the noise source
Final Derivation Steps

- mean squared voltage across C is equivalent to \( \overline{v^2} = \frac{S_v(0)}{4RC} \)
- then, from thermodynamics:
  \[ \overline{v^2} = \frac{S_v(0)}{4RC} - \frac{k_BT}{C} \]
- so, we get:
  \[ S_v(0) = 4k_BT \]
- BUT, what is \( S_v(0) \)? and what’s with the \( (0) \)?

- think Planck blackbody thermal power distribution: \( E(f) \)
  \[ S_v(f) = 4k_BT \]
- for \( hf \ll k_BT \), \( S_v(f) = 4Rk_BT \)
- it’s like the Planck distribution becomes classical \( k_BT \) if \( h \) is small
  - what frequencies correspond to room temperature?
  - \( T=290 \text{ K}; k_BT=0.025 \text{ eV}; f=6 \text{ THz} \) infrared frequencies
- electrical circuits have much lower \( f \) [take as approximation \( f=0 \)...that’s why it’s \( S_v(0) \)]

\[ S_v(0) = 4k_BT \]

Johnson Noise Power

- we see that \( S_v(0) = 4k_BT \) is independent of frequency
- \( S_v(0) \Delta f \) is \( |v_n|^2 \) in the frequency interval \( \Delta f \)
- the open-circuit, root mean square noise voltage is:
  \[ v_n = \sqrt{4k_B T \Delta f} \]
- confirm with some dimensional analysis:
  - blackbody power in a frequency interval: \( E(f) \Delta f \)
  - thermal noise power = \( k_BT \Delta f = v_n^2/4R \) has units of power
  - \( v_n \) has units of voltage
  - frequency-independent, flat power spectrum: it is “white” noise
- flat power spectrum up to infinite frequencies?
  - problem is solved by Planck blackbody formula
  - just like Planck solved the “ultraviolet catastrophe”

Johnson noise is like “blackbody radiation” in a resistor...it is electrical power due to thermal energy
More Physics Behind Johnson Noise

- fluctuation-dissipation theorem in statistical mechanics
  - states that there is a general relationship (a mathematical formula) between physical quantities that behave like fluctuations and physical quantities that behave like dissipation
  - e.g. Brownian motion is related to frictional drag
    - colloidal particles undergo random motion due to molecules of the liquid randomly colliding with them, bumping them around
    - a moving particle through a fluid suffers drag because it is colliding randomly with molecules in the liquid, slowing it down
    - these have the same origin, hence can be related!
    - Einstein-Smoluchowski relation: $D = \mu k_B T$
      - diffusion coefficient is related thermodynamically to mobility (terminal drift velocity / applied force)

- of course, that’s just what Johnson noise is!
  - fluctuation: noise voltage caused by electrons bouncing around
  - dissipation: resistance is drifting electrons getting bumped around
  - ...are thus thermodynamically related
  $$v_n^2 = 4k_B T R \Delta f$$

What are the Numbers Like?

- $T=290$ K, $k_B T = 0.025 \text{ eV}$
- $R=1 \text{ k}\Omega$, $\Delta f = 100 \text{ MHz}$

$$v_n = \sqrt{4k_B T R \Delta f}$$

$$= \sqrt{4(0.025eV)(1000\Omega)(100\times10^6\text{ s}^{-1})}$$

$$e = 1.6\times10^{-19} C; [A] \equiv \frac{C}{s}$$

$$= \sqrt{1.6\times10^{-9}[V^2]}$$
$$= 4\times10^{-5} V \quad \text{or} \quad 40 \mu V$$

**what changes if $\Delta f$ changes?**

thinking about Johnson noise, is there a clever idea that occurs to anyone with respect to instrumentation?  

**can you use Johnson noise to measure something?**